Multiview varieties and reconstruction problems

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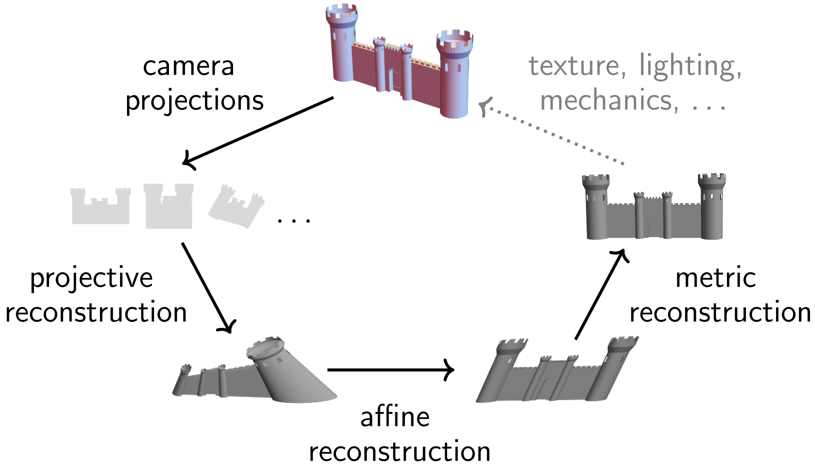
October 6, 2018

Reconstruction problems in computer vision



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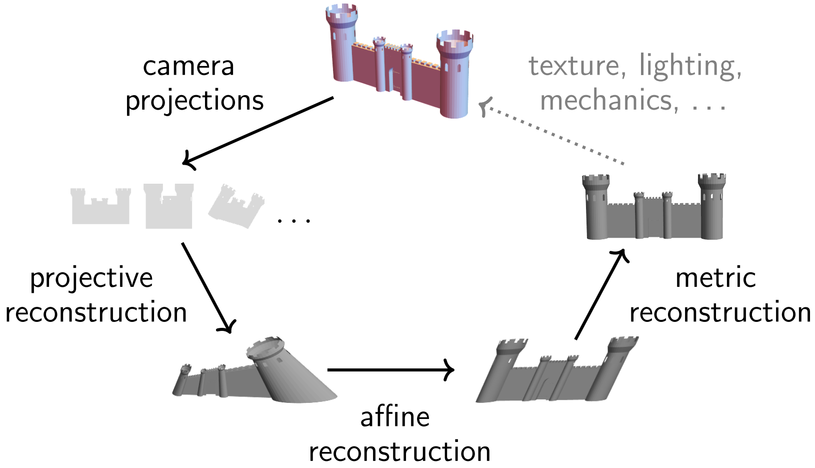
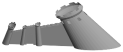
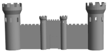


How to describe the problems in arbitrary dimensions?

Model: world spaces

projective	affine	metric
		
$(\mathbb{P}^n, \operatorname{Aut}\mathbb{P}^n)$	$(\mathbb{R}^n, \operatorname{Aff}\mathbb{R}^n)$	$(\mathbb{R}^n, \operatorname{Sim}\mathbb{R}^n)$

- $\mathbb{P}^n$: projective space over $\mathbb{R}$
- $\mathbb{R}^n = \mathbb{P}^n \setminus H$: Euclidean space as a set
- $\operatorname{Aut}\mathbb{P}^n \simeq PGL(n+1)$
- $\operatorname{Aff}\mathbb{R}^n = \{g \in \operatorname{Aut}\mathbb{P}^n \mid g \cdot H \subset H\} \simeq GL(n) \ltimes \mathbb{R}^n$
- $\operatorname{Sim}\mathbb{R}^n = \{g \in \operatorname{Aff}\mathbb{R}^n \mid g \cdot Q \subset Q\} \simeq \mathbb{R}^\times O(n) \ltimes \mathbb{R}^n$

$H \in |\mathcal{O}_{\mathbb{P}^n}(1)|$ — *hyperplane at infinity*

smooth definite $Q \in |\mathcal{O}_H(2)|$ — *absolute quadric*

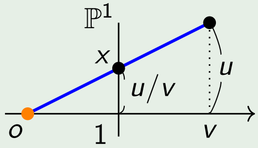
Model: cameras

$s\colon \mathbb{R}^{n+1} \twoheadrightarrow \mathbb{R}^{m+1}$: a surjective linear map $(n > m)$

$l_x \subset \mathbb{R}^{m+1}$: line corresponding to $x \in \mathbb{P}^m$

$\bar{s}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$	— *camera projection*
$\mathbb{P}(\ker s) \subset \mathbb{P}^n$	— *focal locus* (dim : $n - m - 1$)
$\mathbb{P}(s^{-1}(l_x)) \subset \mathbb{P}^n$	— *back-projected plane* (dim : $n - m$)

Example (Pinhole camera model)



$$H = \{w = 0\} \supset Q = \{u^2 + v^2 = 0\}$$

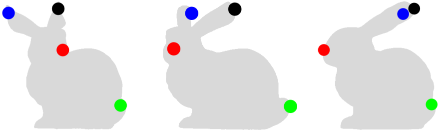
$$\bar{s}\colon \mathbb{P}^2 \ni \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} u \\ v \end{bmatrix} \equiv \begin{bmatrix} u/v \\ 1 \end{bmatrix} \in \mathbb{P}^1$$

$\bar{s}$: *realistic* $\Leftrightarrow$ $n = 3$ and $m = 2$

Additional information for reconstruction

$\bar{s}_i \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{m_i}$ $(i = 1, \ldots, r)$: camera projections

- point correspondences



$\varphi := (\bar{s}_1, \ldots, \bar{s}_r) \colon \mathbb{P}^n \dashrightarrow \prod_i \mathbb{P}^{m_i}$
$\varphi(u) = (\bar{s}_1(u), \ldots, \bar{s}_r(u))$ stands for a correspondence.

- camera motions
- prior knowledge of the scene

Multiview varieties

$X_\varphi = \varphi(\mathbb{P}^n)$ — *multiview variety*
(moduli space of point correspondences)

$\varphi(\mathbb{P}^n) := \overline{\varphi(\mathbb{P}^n \setminus Z)} \subset \prod_i \mathbb{P}^{m_i}$: the image of φ

Z: the union of focal loci $Z_1, \ldots, Z_r$

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Questions

- How to describe X_φ?
- Can we recover φ (up to $\operatorname{Aut} \mathbb{P}^n$) from $X_\varphi \subset \prod_i \mathbb{P}^{m_i}$?

Description 1: via back-projected planes

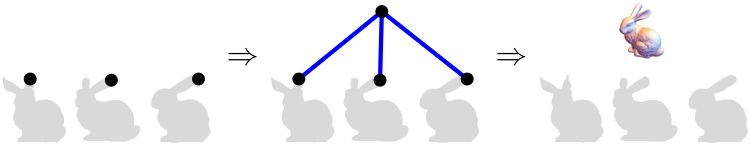
Each $x = (x_1, \ldots, x_r) \in \prod_i \mathbb{P}^{m_i}$ corresponds to an r-tuple of back-projected planes $(P_1, \ldots, P_r)$, where $P_i = \overline{\bar{s}_i^{-1}(x_i)}$.

Description 1

Assume φ is generic.

$$X_\varphi = \{x \in \textstyle\prod_i \mathbb{P}^{m_i} \mid \bigcap_i P_i \neq \varnothing\}.$$

Plotting $\varphi^{-1} \colon X_\varphi \dashrightarrow \mathbb{P}^n$ is refered to as *triangulation*:



Description 2: via Grassmann tensors

Assume φ is generic, and $|\mathbf{m}| := \sum_i m_i > n$.

Write $\mathbf{s} := (s_1, \ldots, s_r)\colon V \hookrightarrow \bigoplus_i W_i$.

Fix $\alpha \in \mathbb{Z}^r$ such that $1 \leq \alpha_i \leq m_i$ and $\sum_i \alpha_i = n+1$.

$\mathrm{pr}_\alpha\colon \bigwedge^{n+1} \bigoplus_i W_i \simeq \bigoplus_\beta \left(\bigotimes_i \bigwedge^{\beta_i} W_i\right) \twoheadrightarrow \bigotimes_i \bigwedge^{\alpha_i} W_i$

$[A^{\sigma_1,\ldots,\sigma_r}] := \left[\mathrm{pr}_\alpha \bigwedge^{n+1} \mathbf{s}(V)\right]$ — *Grassmann tensor* of profile α

$(U_1, \ldots, U_r) \in \prod_i Gr(m_i - \alpha_i, \mathbb{P}(W_i)) \xrightarrow{\prod_i p^i} \prod_i \mathbb{P}(\bigwedge^{\alpha_i} W_i^*)$

Description 2

Assume φ is generic. For any profile α,

$$X_\varphi \cap \prod_i U_i \neq \varnothing \Leftrightarrow \sum_{\sigma_1,\ldots,\sigma_r} A^{\sigma_1,\ldots,\sigma_r} p^1_{\sigma_1} \ldots p^r_{\sigma_r} = 0.$$

Moduli spaces for $|\mathbf{m}| > n$

$$\begin{array}{ccc}
\prod_i \mathbb{P}(V^* \otimes W_i) & & \\
\Big\downarrow{\scriptstyle /\!\!/ PGL(V)} & & \\
\mathrm{Cam} := Gr(n+1, \bigoplus_i W_i) /\!\!/ \mathbb{G}_m^r & \overset{\gamma}{\dashrightarrow} & \mathrm{Hilb}\left(\prod_i \mathbb{P}(W_i)\right) \\
{\scriptstyle \text{Plücker}}\Big\downarrow & \overset{\pi_\alpha}{\searrow} & \\
\mathbb{P}\left(\bigwedge^{n+1} \bigoplus_i W_i\right) /\!\!/ \mathbb{G}_m^r & \underset{\overline{\mathrm{pr}}_\alpha}{\dashrightarrow} & \mathbb{P}\left(\bigotimes_i \bigwedge^{\alpha_i} W_i\right)
\end{array}$$

Cam	moduli space of camera projections
$\mathcal{X} := \gamma(\mathrm{Cam})$	—"— of multiview varieties
$\mathcal{A}_\alpha := \pi_\alpha(\mathrm{Cam})$	—"— of Grassmann tensors of profile α

Examples for realistic cameras:

$\mathcal{A}_{2,2}, \mathcal{A}_{2,1,1}, \mathcal{A}_{1,1,1,1}$: the *epipolar*, *trifocal*, *quadrifocal* varieties

Projective reconstruction theorem

$\mathbf{m} = (m_1, \ldots, m_r)$, $\alpha = (\alpha_1, \ldots, \alpha_r)$

Theorem

Assume φ is generic, and $|\mathbf{m}| > n$.

- *(Hertley–Schaffalitzky '09)*

  If $\mathbf{m} \neq (1^{n+1})$, $\pi_\alpha\colon \mathrm{Cam} \dashrightarrow \mathcal{A}_\alpha$ is generically injective.

  If $\mathbf{m} = (1^{n+1})$, π_α is generically $2:1$.

- *(Aholt–Sturmfels–Thomas '13, Ito–M–Ueda '17+)*

  If $\mathbf{m} \neq (1^{n+1})$, $\gamma\colon \mathrm{Cam} \dashrightarrow \mathcal{X}$ is generically injective.

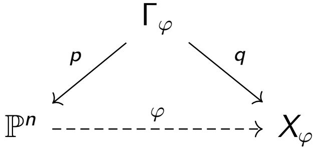
  If $\mathbf{m} = (1^{n+1})$, γ is generically $2:1$.

  If $|\mathbf{m}| \geq 2n-1$, $\mathcal{X}$ is an irreducible component of $\mathrm{Hilb} \prod_i \mathbb{P}^{m_i}$.

Projective reconstruction for $\mathbf{m} \neq (1^{n+1})$

Assume φ is generic, and $|\mathbf{m}| > n$.

$\Gamma_\varphi := (\mathrm{id} \times \varphi)(\mathbb{P}^n) \subset \mathbb{P}^n \times \prod_i \mathbb{P}^{m_i}$: the graph of φ



$$\mathcal{L}_1 = \iota^* \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^{m_1}}(1)$$
$$E_1 = p^{-1}(Z_1)$$
$$D_1 = q(E_1) = \mathbb{P}^{m_1} \times \text{something}$$

- $\mathrm{Sing}\, X_\varphi = \{\dim \bigcap_i P_i \geq 1\}$: singular locus
- X_φ is normal and q is small.
- If $\mathbf{m} \neq (1^{n+1})$, D_1 is uniquely determined only from X_φ.

Theorem (Ito–M–Ueda '17+)

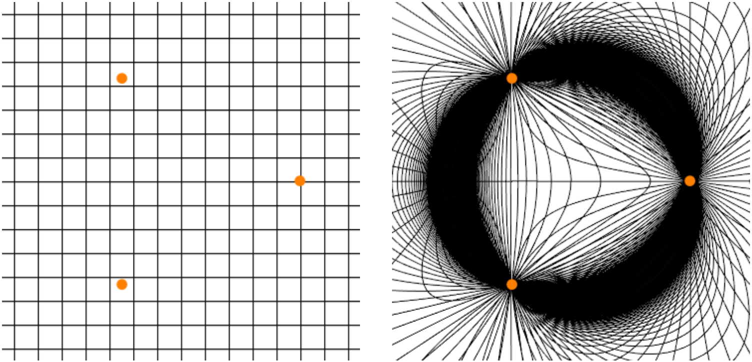
$\varphi^{-1} \colon X_\varphi \dashrightarrow \mathbb{P}^n$ *is given by* $|\mathcal{O}(D_1) \otimes \mathcal{L}_1|$.

Projective reconstruction for $\mathbf{m} = (1^{n+1})$

$\mathbf{s}' \colon \left(\bigoplus_i W_i / \mathbf{s}(V) \right)^* \hookrightarrow \bigoplus_i W_i^*$ gives the dual reconstruction.

Theorem (Ito–M–Ueda '17+)

$\varphi^{-1} \circ \varphi' \in \operatorname{Bir} \mathbb{P}^n$ *is given by* $|\mathcal{O}(n) \otimes I_{Z_1 \cup \cdots \cup Z_{n+1}}|$.



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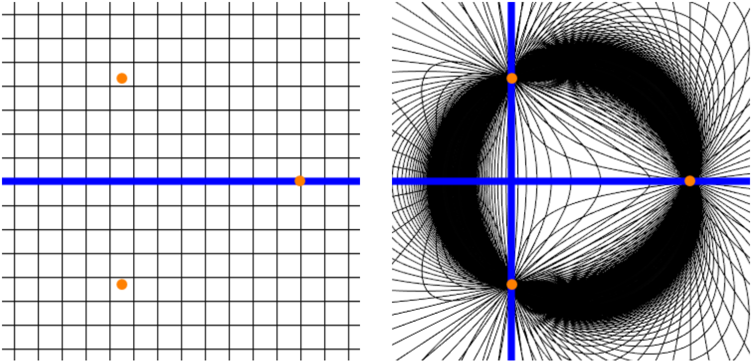

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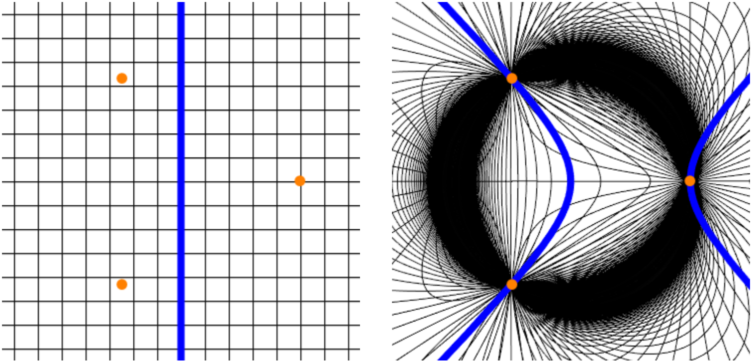

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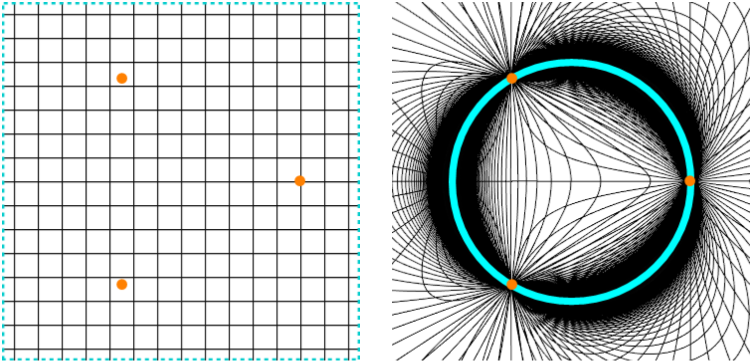


Affine reconstruction theorem

$H \in \mathcal{O}_{\mathbb{P}^n}(1)$: hyperplane at infinity

$T(H) := \{g \in \mathrm{Aut}(\mathbb{P}^n) \mid g = \mathrm{id} \text{ or } (\mathbb{P}^n)^g = H\} \simeq \mathbb{R}^n$
— the group of *pure translations* on $(\mathbb{R}^n = \mathbb{P}^n \setminus H, \mathrm{Aff}\,\mathbb{R}^n)$

$\bar{s}_1, \bar{s}_2$: related by a G-motion $\Leftrightarrow \mathbb{P}^{m_1} = \mathbb{P}^{m_2}$ and $\bar{s}_2 \in \bar{s}_1 G$

$\Delta \subset \mathbb{P}^m \times \mathbb{P}^m$: the diagonal set

Theorem (Ito–M–Ueda, in preparation)

$\varphi = (\bar{s}_1, \bar{s}_2)\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m \times \mathbb{P}^m$: *related by a* $T(H)$*-motion*

Assume $X_\varphi \not\subset \Delta$, *then* H *is uniquely reconstructed from* φ.

The affine structure is compatible with the factoring,

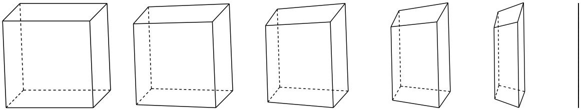
$$\mathbb{P}(V)^n \dashrightarrow \mathbb{P}\left(V/(\ker s_1 \cap \ker s_2)\right)^{m+1} \dashrightarrow \mathbb{P}(W)^m \times \mathbb{P}(W)^m.$$

Summary

- Two formulations for projective reconstruction are obtained for a generic φ and $|\mathbf{m}| > n$.
- Affine reconstruction is described by some degenerate φ and $|\mathbf{m}| \leq n$ in general.

Future topics:

- Degenerate configurations
- Metric reconstruction problems (the difference between $\mathbb{R}$ and $\mathbb{C}$ may give difficulty.)



rotating cube in $\mathbb{R}^4$