Complete intersection Calabi–Yau threefolds in Hibi toric varieties and their smoothing

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Abstract

In this article, we summarize combinatorial description of complete intersection Calabi–Yau threefolds in Hibi toric varieties. Such Calabi–Yau threefolds have at worst conifold singularities, and are often smoothable to non-singular Calabi–Yau threefolds. We focus on such non-singular Calabi–Yau threefolds of Picard number one, and illustrate the calculation of topological invariants, using new motivating examples.

1 Introduction

A *Hibi toric variety* is defined as a projective toric variety $\mathbb{P}_{\Delta(P)}$ associated with an order polytope

$$\Delta(P) = \{(x_u)_{u\in P} \mid 0 \le x_u \le x_v \le 1 \text{ for } u \prec v \in P\} \subset \mathbb{R}^P, \tag{1}$$

for a finite poset $P = (P, \prec)$. For example, all products of projective spaces are Hibi toric varieties; hence at least 2590 topologically distinct non-singular Calabi–Yau threefolds are obtained as complete intersections [12]. In general, complete intersection Calabi–Yau threefolds in Hibi toric varieties have finite number of nodes, and are often smoothable to non-singular Calabi–Yau threefolds by flat deformations. Complete intersections in Grassmannians (or more generally in minuscule Schubert varieties) give basic examples of such smoothing [5, 17].

The purpose of this article is to provide a brief summary on combinatorial descriptions of complete intersection Calabi–Yau threefolds in Hibi toric varieties and their smoothing. Based on [7], we describe the smoothability in terms of posets (Proposition 3.6), and survey the calculation of topological invariants for resulting non-singular simply-connected Calabi–Yau threefolds (Subsection 4.2), by focusing on the case of Picard number one for simplicity. In addition to the summary, we show the simply-connectedness as a corollary of the result on small resolutions for Hibi toric varieties (Proposition 2.6). To illustrate the calculation, we introduce several new examples of such non-singular Calabi–Yau threefolds of Picard number one (Subsection 4.3, Table 1).

A Calabi–Yau threefold is a complex projective threefold X with at worst canonical singularities satisfying $\omega_X \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. There are a huge number of such threefolds, even non-singular. Mirror symmetry is a conjectural duality between a non-singular Calabi–Yau threefold X and another non-singular Calabi–Yau threefold X^*, called a *mirror manifold* for X. Various non-trivial relations between X and X^* are expected. For example, Hodge numbers satisfy

$$h^{i,j}(X) = h^{3-j,i}(X^*) \quad \text{for all } i \text{ and } j. \tag{2}$$

One of the big mysteries of mirror symmetry is whether every non-singular Calabi–Yau threefold X has a mirror manifold X^* or not. Note an obvious exception in the case with $h^{2,1}(X) = 0$, and that the mirror manifold X^* is not unique in general, even as topological manifolds. There is an excellent class of non-singular Calabi–Yau threefolds such that the above question has an affirmative answer; for crepant resolutions of complete intersection Calabi–Yau threefolds in Gorenstein toric Fano varieties, we have mirror manifolds in the same class, called the *Batyrev–Borisov mirrors* [3, 8]. In order to expand this class, the conjectural mirror construction via conifold transitions seems to be a promising direction.

Let X_0 be a Calabi–Yau threefold with finitely many nodes. Suppose that X_0 admits a smoothing $X \rightsquigarrow X_0$ by a flat deformation, and a small resolution $Y \to X_0$. The composite operation connecting two non-singular Calabi–Yau threefolds X and Y is called a *conifold transition*:

$$X \rightsquigarrow X_0 \leftarrow Y. \tag{3}$$

There is a natural closed immersion of the Kuranishi space $\mathrm{Def}(Y)$ to $\mathrm{Def}(X_0)$ [19, Proposition 2.3], and hence, it makes sense to put them together into some giant moduli space. There is a question, commonly referred to as (a version of) *Reid's fantasy*, which asks whether all simply-connected non-singular Calabi–Yau threefolds fit together into a single

irreducible family via conifold transitions [20]. Suppose that X and Y have torsion-free homology for a conifold transition (3). *Morrison's conjecture* in [18] says that the mirror manifolds are also connected via a conifold transition of the opposite direction:

$$Y^* \rightsquigarrow Y_0^* \leftarrow X^*. \tag{4}$$

Together with the spirit of Reid's fantasy, one may expect a mirror construction for a large number of non-singular Calabi–Yau threefolds from the Batyrev–Borisov mirror pairs.

We still do not know the existence of a mirror manifold X^*, even for the smoothing $X \rightsquigarrow X_0$ of a complete intersection Calabi–Yau threefold X_0 in a Hibi toric variety. Nevertheless, we can discuss the mirror symmetry by calculating periods and Picard–Fuchs operators for the conjectural mirror family, as we see in Remark 4.3 for example.

2 Hibi toric varieties

2.1 Examples

Let us begin with simple examples of Hibi toric varieties. For the empty poset, we set the Hibi toric variety $\mathbb{P}_{\Delta(\varnothing)}$ to be a point. For a singleton $u := \{u\}$ (by abuse of notation), the order polytope is a line segment $\Delta(u) = [0, 1]$, and hence, the Hibi toric variety $\mathbb{P}_{\Delta(u)}$ is a projective line $\mathbb{P}^1$.

Let P be a finite poset consisting of $n := |P|$ elements. If P is a *chain*, i.e., a totally ordered set, the order polytope $\Delta(P)$ is a regular simplex defined by the inequalities $0 \le x_1 \le \cdots \le x_n \le 1$, so that the Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is a projective space $\mathbb{P}^n$. It is equally clear the case that P is an *anti-chain*, i.e., the poset in which every pair of elements is incomparable. In this case, the order polytope $\Delta(P)$ is a unit hypercube $[0, 1]^n$, and the Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is the product of n copies of $\mathbb{P}^1$.

Example 2.1. A first non-trivial example is a poset $P = \{u, v, w\}$ with the partial order defined by $u \succ w$ and $v \succ w$. The defining inequalities of the order polytope $\Delta(P)$ is shown in the left of Figure 1, also depicted symbolically in the middle. It becomes a pyramid in $\mathbb{R}^P \simeq \mathbb{R}^3$ as shown in the right of Figure 1. Therefore, the associated Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is a projective cone over $\mathbb{P}^1 \times \mathbb{P}^1$ with a general apex in $\mathbb{P}^3$.

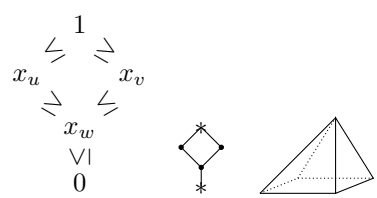


Figure 1: An example of order polytopes

A *disjoint union* $P = P_1 + P_2$ of finite posets P_1 and P_2 is a disjoint union as sets equipped with the partial order $\prec$ satisfying (i) $u \in P_1$, $v \in P_1$ and $u \prec v \in P_1$ imply $u \prec v \in P$, (ii) $u \in P_2$, $v \in P_2$ and $u \prec v \in P_2$ imply $u \prec v \in P$, and (iii) $u \in P_1$ and $v \in P_2$ imply $u \not\sim v \in P$ (i.e., u and v are incomparable in P). The corresponding Hibi toric variety is projectively equivalent to the product of two Hibi toric varieties,

$$\mathbb{P}_{\Delta(P_1+P_2)} \simeq \mathbb{P}_{\Delta(P_1)} \times \mathbb{P}_{\Delta(P_2)}. \tag{5}$$

A *ordinal sum* $P = P_1 \oplus P_2$ of P_1 and P_2 is a disjoint union as sets equipped with the partial order $\prec$ satisfying the same (i) and (ii) as the disjoint union $P_1 + P_2$ above, and (iii)$'$ $u \in P_1$ and $v \in P_2$ imply $u \prec v \in P$. Note that the operation $\oplus$ is not commutative though it is associative. The corresponding Hibi toric variety is a special hyperplane section of a projective join of two Hibi toric varieties with general positions,

$$\mathbb{P}_{\Delta(P_1 \oplus u \oplus P_2)} \simeq \mathrm{Join}\left(\mathbb{P}_{\Delta(P_1)}, \mathbb{P}_{\Delta(P_2)}\right). \tag{6}$$

These operations generalize the examples, a chain $P = \bigoplus_{i=1}^n u_i$, an anti-chain $P = \sum_{i=1}^n u_i$, and $P = w \oplus (u + v) = \varnothing \oplus w \oplus (u + v)$ in Example 2.1.

The posets built up by disjoint unions and ordinal sums from singletons are sometimes called series-parallel posets. One of the simplest examples that are not series-parallel is the poset with the *Hasse diagram*:

(7)

Recall that, in a Hasse diagram for a poset P, a vertex represents an element of P and an oriented edge represents a *covering relation* $u \lessdot v$ on P, that is,

$$u \prec v \text{ and there is no } w \in P \text{ such that } u \prec w \prec v. \tag{8}$$

For example, the Hasse diagram (7) represents the poset $P = \{a, b, c, d\}$ with $a \prec b \succ c \prec d$. The associated Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is a limit of a toric degeneration of a general linear section fourfold in a Grassmannian $G(2, 5)$.

2.2 Invariant subvarieties and singularities

Invariant subvarieties of Hibi toric varieties are again (projectively equivalent to) lower dimensional Hibi toric varieties. We follow the description of invariant subvarieties by Wagner [22].

Let P be a finite poset. The associated bounded poset is defined as

$$\hat{P} := \hat{0} \oplus P \oplus \hat{1}, \tag{9}$$

where $\hat{0}$ and $\hat{1}$ are singletons. By definition, the elements $\hat{0}$ and $\hat{1}$ are the unique minimal and the maximal elements in $\hat{P}$, respectively. Note that the Hasse diagram of $\hat{P}$ can be regarded as the graph describing the defining inequalities of order polytope $\Delta(P)$, as we see in the middle of Figure 1. We use this identification between inequalities with edges, and variables with vertices for the Hasse diagram of $\hat{P}$. Furthermore, by abuse of notation, we write the same symbol P as the Hasse diagram of P. For example, we say that P is *connected* if the Hasse diagram of P is connected, and P is a *cycle* if the Hasse diagram of P is a cycle as an unoriented graph, and so on.

Definition 2.2. Let $\hat{P}$ be a bounded poset. A surjective order-preserving map

$$\varphi \colon \hat{P} \to \hat{P}' \tag{10}$$

with $\varphi(\hat{0}) = \hat{0}$ and $\varphi(\hat{1}) = \hat{1}$ is called a *contraction* of $\hat{P}$ if every fiber is connected and there exists a covering relation $u \lessdot v \in \hat{P}$ for all $\bar{u} \lessdot \bar{v} \in \hat{P}'$ such that $\bar{u} = \varphi(u)$ and $\bar{v} = \varphi(v)$.

There is a one-to-one correspondence between faces of order polytope $\Delta(P)$ and contractions of the associated bounded poset $\hat{P}$. More precisely, a face θ_φ corresponding to a contraction $\varphi \colon \hat{P} \to \hat{P}'$ is unimodular equivalent to the order polytope $\Delta(P')$. In other words, the associated invariant subvariety of a Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is projectively equivalent to the Hibi toric variety $\mathbb{P}_{\Delta(P')}$, as mentioned at the beginning of this subsection. In particular, we have one-to-one correspondences between facets of $\Delta(P)$ and edges of $\hat{P}$, and vertices of $\Delta(P)$ and *order ideals* of P. Here an order ideal is defined as a subset $\tau \subset P$ satisfying

$$u \in \tau, v \in P \text{ and } u \succ v \text{ imply } v \in \tau. \tag{11}$$

Let us write the set of edges of $\hat{P}$ as $E = \mathrm{Edges}(\hat{P})$, and the set of order ideals of P as $J(P)$.

We illustrate the correspondences by using the poset P in Example 2.1. We have five facets corresponding to E, and five vertices corresponding to $J(P)$. The defining equalities of a face θ_φ can be obtained by making all variables in a fiber $\varphi^{-1}(\bar{u})$ equal.

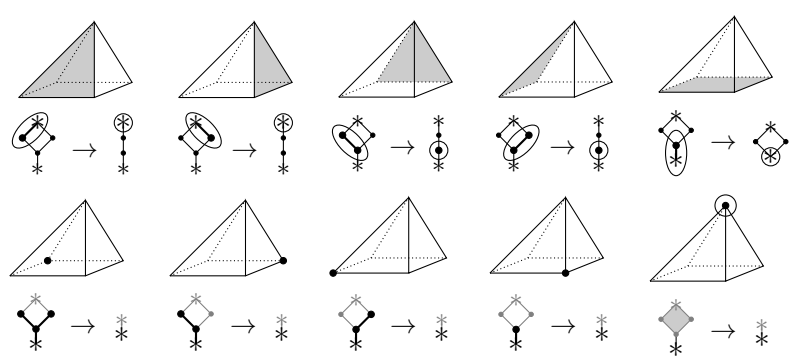


Figure 2: One-to-one correspondence between faces and contractions

In Figure 2, four facets of the order polytope $\Delta(P)$ are meeting at the same vertex circled. Hence the corresponding point should be singular in $\mathbb{P}_{\Delta(P)}$. In general, a singular locus comes from a contraction replacing more inequalities to equalities than codimension. A subposet $C \subset \hat{P}$ is said to be *convex* if it satisfies

$$u \in C, v \in C \text{ and } u \prec w \prec v \text{ imply } w \in C. \tag{12}$$

Furthermore, let us call $C \subset \hat{P}$ a *minimal convex cycle* if C is (i) a full subposet not containing both $\hat{0}$ and $\hat{1}$, and (ii) a convex cycle such that all convex full subposets $C' \subset C$ are trees.

Theorem 2.3 ([22, Corollary 2.4])**.** *Let P be a finite poset. An irreducible singular locus of $\mathbb{P}_{\Delta(P)}$ corresponds to a minimal convex cycle $C \subset \hat{P}$. For the corresponding contraction, all the fibers are singletons except one fiber C.*

Remark 2.4. Now it is worth noting the homogeneous coordinate rings of Hibi toric varieties. Since all the lattice points in $\Delta(P)$ are vertices, the homogeneous coordinate ring of $\mathbb{P}_{\Delta(P)}$ is a *Hibi algebra*,

$$A_{J(P)} = \mathbb{C}\left[J(P)\right]/I_{J(P)}, \tag{13}$$

where $\mathbb{C}\left[J(P)\right]$ is the polynomial $\mathbb{C}$-algebra in variables p_τ for $\tau \in J(P)$, and $I_{J(P)}$ is the ideal coming from linear relations of vertices of $\Delta(P)$. In fact, the ideal $I_{J(P)}$ has quadratic generators,

$$p_\alpha p_\beta - p_{\alpha\vee\beta} p_{\alpha\wedge\beta} \text{ for all } \alpha \not\sim \beta \in J(P). \tag{14}$$

Note here that $J(P)$ is a *lattice*, i.e., a poset with the least upper bound $\alpha \vee \beta$ and the greatest lower bound $\alpha \wedge \beta$ for each pair of elements. In fact, $J(P)$ with the partial order given by set inclusions is equipped with $\alpha \vee \beta = \alpha \cup \beta$ and $\alpha \wedge \beta = \alpha \cap \beta$. Furthermore, $J(P)$ becomes a *distributive lattice*, i.e., the lattice with distributive laws,

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \quad \text{for all } \alpha, \beta, \gamma \in J(P). \tag{15}$$

One may start from finite distributive lattices instead of finite posets, which gives another description of Hibi toric varieties in literature.

Example 2.5. Under the notation in Remark 2.4, the Hibi toric variety $\mathbb{P}_{\Delta(P)}$ in Example 2.1 is embedded as a quadric threefold in $\mathbb{P}^4$ defined by

$$p_{\nwarrow} p_{\swarrow} - p_{\vee} p_{\cdot} = 0. \tag{16}$$

2.3 Divisors

Let P be a finite poset. For each edge $e \in E = \mathrm{Edges}(\hat{P})$, we have the corresponding invariant prime divisor, denoted by D_e. We write $D_{E'} = \sum_{e \in E'} D_e$ for each subset $E' \subset E$. The linear equivalences in the divisor class group $\mathrm{Cl}(\mathbb{P}_{\Delta(P)})$ are generated by the following relations:

$$\sum_{s(e)=u} D_e \simeq \sum_{t(e')=u} D_{e'} \quad \text{for all } u \in P, \tag{17}$$

where we write $t(e) \lessdot s(e)$ for each $e = (s(e), t(e)) \in E \subset \hat{P} \times \hat{P}$.

Let us describe the Picard group of $\mathbb{P}_{\Delta(P)}$. First, suppose P is connected. For an order ideal $\tau \subset P$ and a subset $E' \subset E$, a set of edges

$$E'(\tau) = \left\{ e \in E' \mid t(e) \in \hat{0} \oplus \tau \text{ and } s(e) \notin \hat{0} \oplus \tau \right\} \tag{18}$$

defines a Weil divisor $D_{E'(\tau)}$ on $\mathbb{P}_{\Delta(P)}$. We have $D_{E(\tau)} \simeq D_{E(P)}$ for all $\tau \in J(P)$. The divisor $D_{E(P)}$ is, in fact, the Cartier divisor corresponding to the lattice polytope $\Delta(P)$ itself. One can show that the Picard group is isomorphic to $\mathbb{Z}$ generated by the associated very ample invertible sheaf $\mathcal{O}(1) = \mathcal{O}(D_{E(P)})$.

More generally, suppose $P = \sum_{j=1}^{\rho} P_j$ with ρ connected components $P_1, \ldots, P_\rho$. Note that there is a natural decomposition as sets

$$E = \bigsqcup_{j=1}^{\rho} E_j, \tag{19}$$

where $E_j := \mathrm{Edges}(\hat{P}_j) \subset E$ for each j. The Picard group $\mathrm{Pic}(\mathbb{P}_{\Delta(P)})$ becomes a free abelian group of rank ρ generated by

$$\mathcal{L}_j := \mathcal{O}\left(D_{E_j(P_j)}\right) \tag{20}$$

for each connected component $P_j \subset P$. We have

$$\mathcal{O}(1) = \mathcal{O}(D_{E(P)}) = \bigotimes_{j=1}^{\rho} \mathcal{L}_j. \tag{21}$$

Next, we note a formula for the self-intersection number,

$$\left(D_{E(P)}^n\right) = n! \operatorname{Vol} \Delta(P) = c_{J(P)}, \tag{22}$$

where $c_{J(P)}$ denotes the number of maximal chains on $J(P)$. It follows from a formula for the Hilbert–Poincaré series of Hibi algebra $A_{J(P)}$ obtained by [15, Corollary of Lemma 5].

Lastly, let us suppose P is *pure*. Recall that a finite poset P is called pure if every maximal chain on P has the same length. We define a height $h(u)$ of $u \in \hat{P}$ as the length of the longest chain bounded above by u in $\hat{P}$, and write $h_P = h(\hat{1})$. Thus an anti-canonical divisor $-K_{\mathbb{P}_{\Delta(P)}} = D_E$ is written as

$$-K_{\mathbb{P}_{\Delta(P)}} = \sum_{k=1}^{h_P} D_{E(\tau_k)} \simeq h_P D_{E(P)}, \tag{23}$$

where $\tau_k := \{u \in P \mid h(u) < k\} \in J(P)$ for $k = 1, \ldots, h_P$. Together with (21), it turns out that the Hibi toric variety $\mathbb{P}_{\Delta(P)}$ for a pure poset P is a Gorenstein Fano variety with $\omega^\vee = \mathcal{O}(-K_{\mathbb{P}_{\Delta(P)}}) \simeq \mathcal{O}(h_P)$. Moreover, one can show that it has at worst terminal singularities, by looking at the normal fan Σ of $\Delta(P)$ (see [13, Lemma 1.4]).

2.4 Small resolution

Let P be a finite pure poset. The associated Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is a Gorenstein terminal Fano variety with $\omega^\vee \simeq \mathcal{O}(h_P)$. If $\mathbb{P}_{\Delta(P)}$ is $\mathbb{Q}$-factorial in addition, it turns out to be non-singular, and even more, a product of projective spaces by [13, Corollary 2.4]. Even if it is not $\mathbb{Q}$-factorial, we have the following property indicating the mildness of singularities of $\mathbb{P}_{\Delta(P)}$.

Proposition 2.6. *For a finite pure poset P, any toric crepant $\mathbb{Q}$-factorialization of the Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is a small resolution.*

Proof. Let P be a finite poset, $N = \mathbb{Z}P$ and $M = \mathbb{Z}^P$ the free abelian groups of rank $n = |P|$ dual to each other, and $N_\mathbb{R} = \mathbb{R}P$ and $M_\mathbb{R} = \mathbb{R}^P$ the real scalar extensions, respectively.

First, let us see a description of the normal fan Σ in $N_\mathbb{R}$ for the order polytope $\Delta(P) \subset M_\mathbb{R}$. By definition, a one-dimensional cone in Σ is generated by the normal vector of a facet of $\Delta(P)$. Hence it corresponds to an edge of $\hat{P}$. Let $\delta(e) \in N$ denote such primitive vector associated with $e \in E$, The map δ is extended to be the composite linear map $\delta = \mathrm{pr} \circ \partial \colon \mathbb{Z}E \to N$ of

$$\partial \colon \mathbb{Z}E \to \mathbb{Z}\hat{P} = N \oplus \mathbb{Z}\hat{0} \oplus \mathbb{Z}\hat{1}, \quad e \mapsto \partial(e) := t(e) - s(e) \tag{24}$$

and a projection $\mathrm{pr} \colon N \oplus \mathbb{Z}\hat{0} \oplus \mathbb{Z}\hat{1} \to N$. By using the same symbol δ as the real extension, each maximal dimensional cone in Σ associated with an order ideal $\tau \in J(P)$ is written as

$$\sigma_\tau := \operatorname{Cone} \delta\left(E \setminus E(\tau)\right) \subset N_\mathbb{R}, \tag{25}$$

where $E(\tau)$ is defined by (18). On the other hand, $\operatorname{Conv} \delta(E)$ is a Gorenstein terminal Fano polytope by [13, Lemma 1.3–1.5]. Namely, for any $\tau \in J(P)$, all the primitive generators of the cone σ_τ, i.e., the elements in $\delta(E \setminus E(\tau))$, lie on an affine hyperplane with integral distance one from the origin, and it holds

$$\left(\operatorname{Conv} \delta(E \setminus E(\tau))\right) \cap N = \delta\left(E \setminus E(\tau)\right). \tag{26}$$

Suppose P is pure, and let $X_{\widehat{\Sigma}} \to \mathbb{P}_{\Delta(P)}$ be a toric crepant $\mathbb{Q}$-factorialization. In other words, $\widehat{\Sigma}$ is a maximal simplicial refinement of the normal fan Σ of $\Delta(P)$ such that $X_{\widehat{\Sigma}}$ denotes the corresponding $\mathbb{Q}$-factorial toric variety. Since $\mathbb{P}_{\Delta(P)}$

has at worst terminal singularities, the crepant birational morphism $X_{\widehat{\Sigma}} \to \mathbb{P}_{\Delta(P)}$ is a small modification by definition. Hence it is sufficient to show that $X_{\widehat{\Sigma}}$ is non-singular.

Fix a maximal dimensional cone σ in $\widehat{\Sigma}$. Since (25) and (26), there exist an order ideal $\tau \subset P$ and a subset $B \subset E \setminus E(\tau)$ consisting of $n+1$ elements such that $\sigma = \operatorname{Cone} \delta(B) \subset \sigma_\tau$. As in the example shown in Figure 3, the subgraph $(\hat{P}, E \setminus E(\tau))$ of the Hasse diagram of $\hat{P}$ defining σ_τ consists of two connected graphs, and the subgraph $(\hat{P}, B)$ defining σ consists of two connected tree graphs. In fact, if $(\hat{P}, B)$ contains a cycle, σ cannot have maximal dimension. Therefore, we have a unique unoriented path in $(\hat{P}, B)$ from any $u \in P$ to $\hat{0}$ or $\hat{1}$, which attains a value $\pm u \in \mathbb{Z}\delta(B)$ by summing up and mapping by δ. Hence $\delta(B)$ forms a $\mathbb{Z}$-basis of $N = \mathbb{Z}P$. Since σ is arbitrary, it follows that $X_{\widehat{\Sigma}}$ is non-singular.

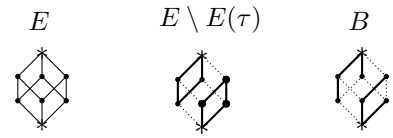


Figure 3: An example of subgraphs corresponding to σ_τ and σ

□

3 CICY threefolds in Hibi toric varieties

3.1 Examples

We describe Calabi–Yau threefolds obtained as a complete intersection of general sections of invertible sheaves in Hibi toric varieties. We call such Calabi–Yau threefolds *complete intersection Calabi–Yau (CICY)* threefolds in Hibi toric varieties.

Let P be a finite poset, and X_0 a CICY threefold in $\mathbb{P}_{\Delta(P)}$. From the adjunction formula, $\mathbb{P}_{\Delta(P)}$ has at worst Gorenstein singularities. In other words, all connected components of P are pure. If P is a disjoint union of several pure connected posets, we have a number of Calabi–Yau threefolds as complete intersections of nef divisors in $\mathbb{P}_{\Delta(P)}$, e.g., complete intersection Calabi–Yau threefolds in products of projective spaces. However, we assume in the sequel that P is pure connected for simplicity. Under this assumption, X_0 is merely a complete intersection of very ample divisors in $\mathbb{P}_{\Delta(P)}$. Let $(d_1, \ldots, d_r) \subset \mathbb{P}_{\Delta(P)}$ denote a complete intersection variety of very ample divisors defined by general sections of $\mathcal{O}(d_1), \ldots, \mathcal{O}(d_r)$, respectively. Then $X_0 = (d_1, \ldots, d_r) \subset \mathbb{P}_{\Delta(P)}$ is a CICY threefold if and only if

$$\sum_{j=1}^{r} d_j = h_P \quad \text{and} \quad |P| - r = 3. \tag{27}$$

Example 3.1. The poset in (7) gives an example of hypersurface Calabi–Yau threefolds in Hibi toric varieties. Thus $X_0 = (d_1 = 3) \subset \mathbb{P}_{\Delta(P)}$ in this case.

Example 3.2. As an example to illustrate calculations, we introduce a finite pure connected poset $P = \mathcal{P}_1$:

(28)

We have $|\mathcal{P}_1| = 6$ and $h_{\mathcal{P}_1} = 3$, and hence, the associated Hibi toric variety $\mathbb{P}_{\Delta(\mathcal{P}_1)}$ is a six-dimensional Gorenstein terminal Fano variety with $\omega^\vee \simeq \mathcal{O}(3)$. We have a linear section Calabi–Yau threefold $X_0 = (1^3) \subset \mathbb{P}_{\Delta(\mathcal{P}_1)}$.

The first part of $J(\mathcal{P}_1)$ corresponds to order ideals in the left of Figure 4. By continuing while focusing on set inclusions, we obtain the lattice $J(\mathcal{P}_1)$ as in the middle of Figure 4, consisting of $|J(\mathcal{P}_1)| = 18$ elements. Moreover, we have $c_{J(\mathcal{P}_1)} = 48$, the number of maximal chains on $J(\mathcal{P}_1)$, by counting as in the right in Figure 4.

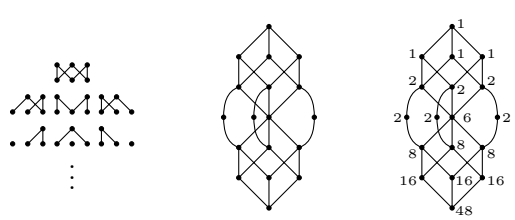


Figure 4: On the Hasse diagram of $J(\mathcal{P}_1)$

3.2 Stringy Hodge numbers

Let P be a pure connected poset and $X_0 = (d_1, \dots, d_r) \subset \mathbb{P}_{\Delta(P)}$. We have a small resolution $Y \to X_0$, by taking the strict transform of X_0 for a small toric resolution $X_{\widehat{\Sigma}} \to \mathbb{P}_{\Delta(P)}$ for example. In this case, the stringy Hodge numbers of X_0 are nothing but usual Hodge numbers of Y. From [4, Proposition 8.6], the following combinatorial formulas hold.

Proposition 3.3.

$$h^{1,1}_{\mathrm{st}}(X_0) = h^{1,1}(Y) = |E| - |P|, \tag{29}$$

$$\begin{aligned} h^{1,2}_{\mathrm{st}}(X_0) = h^{1,2}(Y) = &\sum_{i\in[r]} \left[\sum_{J\subset[r]} (-1)^{|J|} l\left((d_i - d_J)\Delta(P)\right) \right] \\ &- \sum_{J\subset[r]} (-1)^{r-|J|} \left[\sum_{e\in E} l^*(d_J\theta_e) \right] - |P|, \end{aligned} \tag{30}$$

where $l(\theta)$ and $l^*(\theta)$ denote the number of lattice points in a face $\theta \subset M_{\mathbb{R}}$ and in the interior of θ, respectively; $[r] = \{1, \dots, r\}$, $d_J = \sum_{j\in J} d_j$ and θ_e is the facet of $\Delta(P)$ corresponding to an edge $e \in E$.

Note that a nonzero contribution in the first term of (30) comes only from the range of $d_i - d_J \geq 0$, and in the second term it comes only from the range $d_J = h_P - 1$ or h_P. In particular, if (i) $d_j = 1$ for all j, and (ii) P has no ordinal summand of singleton, i.e., $P \neq P_1 \oplus u \oplus P_2$ for any P_1 and P_2, we have

$$h^{1,2}_{\mathrm{st}}(X_0) = h^{1,2}(Y) = h_P\left(|J(P)| - h_P\right) - \sum_{e\in E} l^*\left(h_P\theta_e\right) - |P|. \tag{31}$$

Example 3.4. For the example $P = \mathcal{P}_1$ and a complete intersection $X_0 = (1^3)$, we obtain $h^{1,1}_{\mathrm{st}}(X_0) = 12 - 6 = 6$ from (29). Since the both conditions (i) and (ii) are satisfied, the simplified formula (31) holds in this case. By the symmetry $\mathfrak{S}_3 \times \mathfrak{S}_3$ of $\hat{\mathcal{P}}_1$ as a poset and the order duality, it suffices to see the following two types of facets:

(32)

There are six facets for each type, and clearly $l^*(3\theta) = 1$ (resp. 0) for the former (resp. the latter) type. Therefore $h^{1,2}_{\mathrm{st}}(X_0) = 3(18 - 3) - 6 - 6 = 33$.

3.3 Numbers of nodes

Recall that three-dimensional Gorenstein terminal toric singularities are at worst nodes (i.e, ordinary double points), since they are presented by three-dimensional cones over a unit triangle or a unit square. Together with the Bertini-type theorem for toroidal singularities, the singularities of X_0 are also at worst nodes. We count the number of nodes $\mathrm{dp}(X_0)$ on X_0 in the following.

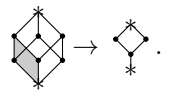
Each node on X_0 lies on one singular locus of codimension three of $\mathbb{P}_{\Delta(P)}$, corresponding to a minimal convex cycle $C \subset \hat{P}$ with four elements. There are four types of such minimal convex cycles:

(33)

Let $\Lambda_4(\hat{P})$ denote the set of such minimal convex cycles consisting of four elements on $\hat{P}$. For each $C \in \Lambda_4(\hat{P})$, there is the contraction $\hat{P} \to \hat{P}_C$ such that all the fibers are singleton except one fiber C. Of course it holds $|P_C| = |P| - 3$ for all $C \in \Lambda_4(\hat{P})$. Hence C defines a singular locus of codimension three and of degree $\deg \Delta(P_C) = c_{J(P_C)}$ from (22). Therefore, the number of nodes $\mathrm{dp}(X_0)$ is calculated by a formula

$$\mathrm{dp}(X_0) = \prod_{j=1}^{r} d_j \sum_{C\in\Lambda_4(\hat{P})} c_{J(P_C)}. \tag{34}$$

Example 3.5. There are six minimal convex cycles on $\hat{\mathcal{P}}_1$, each of which consists of four elements. By symmetry, they are all equivalent to



(35)

Since the locus is a quadric threefold in Example 2.5, we obtain $\mathrm{dp}(X_0) = 6 \cdot 2 = 12$.

3.4 Smoothability

For smoothability, we follow the argument in the case of hypersurfaces in toric varieties by [7]. Let $\{p_1, \ldots, p_{\mathrm{dp}}\}$ be the set of nodes on X_0, where $\mathrm{dp} = \mathrm{dp}(X_0)$, and $f : Y \to X_0$ be a small resolution. The exceptional lines $L_i := f^{-1}(p_i) \simeq \mathbb{P}^1$ for $i = 1, \ldots, \mathrm{dp}$ form a linear subspace of $H_2(Y, \mathbb{C})$. By [19, Theorem 2.5], the Calabi–Yau threefold X_0 is smoothable by a flat deformation if and only if the homology classes $[L_i] \in H_2(Y, \mathbb{C})$ satisfy a relation,

$$\sum_{i=1}^{\mathrm{dp}} \alpha_i [L_i] = 0, \tag{36}$$

where $\alpha_i \neq 0$ for all $i = 1, \ldots, \mathrm{dp}$.

Note that one can identify

$$H_2(Y, \mathbb{Q}) \simeq \left\{ (\lambda_e)_{e \in E} \,\middle|\, \sum_{e \in E} \lambda_e \delta(e) = 0 \right\} \subset \mathbb{Q}^E. \tag{37}$$

Under this identification, the homology class $[L_i]$ coincides with a relation,

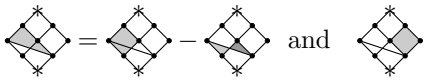
$$\rho_C \ : \ \delta(e_p) + \delta(e_q) - \delta(e_r) - \delta(e_s) = 0 \tag{38}$$

up to signs, where the corresponding node p_i lies on a singular locus associated with a minimal convex cycle $C \in \Lambda_4(\hat{P})$, and the cycle C with an orientation passes through the four edges; e_p, e_q in the forward direction and e_r, e_s in the opposite direction.

Proposition 3.6. *Let P be a pure connected poset, X_0 a CICY threefold of degree $(d_1, \ldots, d_r)$ in the Hibi toric variety $\mathbb{P}_{\Delta(P)}$. If $\prod_{j=1}^r d_j > 1$, X_0 is smoothable. If $\prod_{j=1}^r d_j = 1$, X_0 is smoothable if and only if for any $C \in \Lambda_4(\hat{P})$ such that P_C is a chain the element ρ_C is a linear combination of the remaining elements $\rho_{C'}$ with $C' \in \Lambda_4(\hat{P})$, $C \neq C'$.*

Example 3.7. For the example $P = \mathcal{P}_1$, $X_0 = (1^3)$ is smoothable since P_C is not a chain for all $C \in \Lambda_4(\hat{\mathcal{P}}_1)$ as we see in (35), although $\prod_{j=1}^3 d_j = 1$.

Example 3.8. There are cases that X_0 is not smoothable. For example, in (39) we present the two cycles on the depicted $\hat{P}$ satisfying the condition that P_C is a chain,



(39)

The former cycle is a linear combination of remaining cycles as expressed by abuse of notation. However, the latter cycle is linearly independent to other cycles. Therefore, $X_0 = (1^4)$ is not smoothable by Proposition 3.6.

4 Smoothing of CICY threefolds in Hibi toric varieties

4.1 Simply-connectedness

Proposition 4.1 (Corollary of Proposition 2.6)**.** *Let P be a pure poset, X_0 a CICY threefold in $\mathbb{P}_{\Delta(P)}$, and $Y \to X_0$ a small resolution. Then Y is simply-connected. If a smoothing $X \rightsquigarrow X_0$ exists, X is also simply-connected.*

Proof. By Proposition 2.6, we have a small resolution $X_{\widehat{\Sigma}} \to X_\Sigma = \mathbb{P}_{\Delta(P)}$. Since $X_{\widehat{\Sigma}}$ is a compact toric variety, it is simply-connected (see for example [10, Theorem 9.1]). Let $\Sigma^{(1)}$ denote the subfan of Σ consisting of cones of dimension less than or equal to one. Note $\widehat{\Sigma}^{(1)} = \Sigma^{(1)}$. The quasi-projective toric variety $X_{\Sigma^{(1)}}$ is simply-connected as well. In fact, the difference between $X_{\widehat{\Sigma}}$ and $X_{\Sigma^{(1)}}$ in subsets of real codimension four does not effect fundamental groups. By the Lefschetz theorem for non-singular quasi-projective manifolds [11, 14], a complete intersection $X_{\Sigma^{(1)}} \cap X_0 = X_{\widehat{\Sigma}^{(1)}} \cap Y$ is also simply-connected. Similarly as above, the difference between $X_{\widehat{\Sigma}^{(1)}} \cap Y$ and Y does not effect fundamental groups. Therefore Y is also simply-connected. The latter statement follows from the fact that a conifold transition does not change fundamental groups. □

Let X be a smoothing of a CICY threefold in Hibi toric variety. We do not know whether homology groups of X can have torsion or not. Suppose that X has torsion-free homology and $h^{1,1}(X) = 1$. In this case, by Wall's theorem [23, Theorem 5], the diffeomorphism class of X is determined only by the three topological invariants,

$$\deg X,\ c_2(X) \cdot H, \quad \text{and} \quad \chi(X), \tag{40}$$

where H is the hyperplane class, $c_2(X)$ is the second Chern class and $\chi(X) = 2\left(h^{1,1}(X) - h^{2,1}(X)\right)$ is the topological Euler number of X. We summarize the calculation of these topological invariants in the next subsection.

4.2 Topological invariants

For a conifold transition $X \rightsquigarrow X_0 \leftarrow Y$, Hodge numbers satisfy

$$h^{1,1}(X) = h^{1,1}(Y) - \mathrm{rk}, \tag{41}$$

$$h^{1,2}(X) = h^{1,2}(Y) + \mathrm{dp} - \mathrm{rk}, \tag{42}$$

where $\mathrm{rk} = \mathrm{rk}(X_0)$ is the dimension of linear subspace of $H_2(Y, \mathbb{C})$ spanned by classes of exceptional lines $[L_i]$ for $i = 1, \dots \mathrm{dp}$, i.e.,

$$\mathrm{rk}(X_0) = \mathrm{rank}\left(\rho_C \,\middle|\, C \in \Lambda_4(\hat{P})\right). \tag{43}$$

and $\mathrm{dp} = \mathrm{dp}(X_0)$ is the number of nodes on X_0, which we compute by (34). In particular, from $h^{1,1}(Y) = |E| - |P| = b_1(\hat{P}) + 1$, we have $h^{1,1}(X) = 1$ if and only if all minimal cycles on $\hat{P}$ are generated by cycles in $\Lambda_4(\hat{P})$.

Assume $h^{1,1}(X) = 1$. From (22) and the invariance by a flat deformation, it holds

$$\deg X = \deg X_0 = c_{J(P)} \prod_{j=1}^{r} d_j. \tag{44}$$

Since also the invariance $\chi(X, \mathcal{O}_X(1)) = \chi(X_0, \mathcal{O}_{X_0}(1))$ and a standard cohomology calculation for complete intersection varieties in $\mathbb{P}_{\Delta(P)}$, we obtain

$$\chi(X, \mathcal{O}_X(1)) = \dim H^0(X_0, \mathcal{O}_{X_0}(1)) \tag{45}$$

$$= \dim H^0(\mathbb{P}_{\Delta(P)}, \mathcal{O}_{\mathbb{P}_{\Delta(P)}}(1)) - r_1 \tag{46}$$

$$= |J(P)| - r_1, \tag{47}$$

where $r_1 = \#\{j \mid d_j = 1\}$. Therefore the Hirzebruch–Riemann–Roch theorem gives

$$c_2(X) \cdot H = 12\chi(X, \mathcal{O}_X(1)) - 2\deg X \tag{48}$$

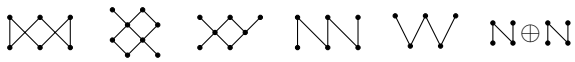
$$= 12\left(|J(P)| - r_1\right) - 2c_{J(P)} \prod_{j=1}^{r} d_j. \tag{49}$$

Example 4.2. For $P = \mathcal{P}_1$, $h^{1,1}(X) = 1$ holds since all minimal cycles are in $\Lambda_4(\hat{\mathcal{P}}_1)$. Hence $\mathrm{rk}(X_0) = 5$ by (41). From $h^{1,2}(Y) = 33$ and $\mathrm{dp}(X_0) = 12$, it holds $h^{1,2}(X) = 40$ and $\chi(X) = 2(h^{1,1}(X) - h^{1,2}(X)) = -78$ by (42). We also obtain $\deg(X) = 48$ and $c_2(X) \cdot H = 12(18 - 3) - 2 \cdot 48 = 84$ by (44) and (49), respectively.

4.3 Examples

Let P be a pure poset and X_0 a CICY threefold in $\mathbb{P}_{\Delta(P)}$. We write $X = X_P$ for a smoothing $X \rightsquigarrow X_0$ if it exists. In spite of the large number of smoothable CICY threefolds in Hibi toric varieties, there are few examples of X_P with $h^{1,1}(X_P) = 1$. From [17, Proposition 3.1], we have twelve such threefolds as complete intersections in minuscule Schubert varieties up to deformation equivalence; five in projective spaces, five in other Grassmannians $G(k, n)$, one in an orthogonal Grassmannian $OG(5, 10)$, and one in a singular Schubert variety of the Cayley plane $\mathbb{OP}^2$. The latter two threefolds are also regarded as complete intersections of some homogeneous vector bundles on Grassmannians, i.e., No. 4 and No. 7 in [16, Table 1], respectively. Apart from these twelve examples, we introduce six more examples in Table 1. The topological invariants are computed by formulas in the previous subsection.

Table 1: Examples of X of Picard number one.

posets $\mathcal{P}_i$	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	$\mathcal{P}_6$
Hasse diagrams						
degrees	(1^3)	(1^5)	(1^4)	(1^3)	$(1, 2)$	(1^5)
$\deg X_{\mathcal{P}_i}$	48	29	42	61	32	25
$c_2(X_{\mathcal{P}_i}) \cdot H$	84	74	84	94	80	70
$\chi(X_{\mathcal{P}_i})$	-78	-100	-96	-86	-116	-100

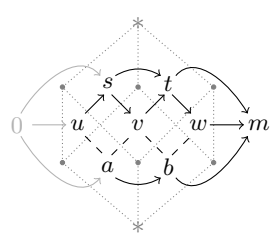
Note that $X_{\mathcal{P}_4}$, $X_{\mathcal{P}_5}$ and $X_{\mathcal{P}_6}$ are deformation equivalent to complete intersections of some homogeneous vector bundles on Grassmannians, i.e., No. 23, No. 20 and No. 5 in [16, Table 1], respectively (from a private communication with Daisuke Inoue and Atsushi Ito for $X_{\mathcal{P}_4}$). Furthermore, $X_{\mathcal{P}_5}$ and $X_{\mathcal{P}_6}$ are also regarded as a complete intersection $(1^2, 2)$ in a Lagrangian Grassmannian $LG(3, 6)$, and a complete intersection of two Grassmannians $G(2, 5)$ in $\mathbb{P}^9$, respectively.

Remark 4.3. Let us discuss mirror symmetry for examples in Table 1. For each conifold transition $X \rightsquigarrow X_0 \leftarrow Y$, we expect another conifold transition $Y^* \rightsquigarrow Y_0^* \leftarrow X^*$ in the mirror side, if X and Y have torsion-free homology. We have a Batyrev–Borisov mirror Y^* for $Y \subset \mathbb{P}_{\widehat{\Sigma}}$, and the degeneration $Y^* \rightsquigarrow Y_0^*$ corresponding to the small resolution $Y \to X_0$ based on the argument by [2]. However, we do not know whether Y_0^* has the same number of nodes as X_0 and admits a small resolution $X^* \to Y_0^*$ or not. In spite of that, periods and the Picard–Fuchs operator vanishing the periods for the conjectural mirror family are computable in advance. The resulting operators in the case of $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$ and $\mathcal{P}_6$ coincide with already known operators, #195, #28, #124, #42, and #101 in [21, 1], respectively.

The operator for $\mathcal{P}_1$ seems unknown, thus we write it here. A formula for the fundamental period $\omega_0(z) = \sum_{m=0}^{\infty} A_m z^m$ is given in [17, Eq. (5,2)], where

$$A_m = \sum_{s,t,u,v,w} \binom{s}{u}\binom{v}{s}\binom{t}{s}\binom{t}{v}\binom{w}{t}\binom{m}{t}\binom{m}{w}\binom{v-t+w}{u-s+v}\binom{m}{v-t+w} \tag{50}$$

is read from the (slightly modified) dual graph of $\hat{\mathcal{P}}_1$, by associating binomial coefficients with oriented edges and linear relations with pairs of dashed edges:



$$\begin{cases} a = u - s + v, \\ b = v - t + w. \end{cases} \tag{51}$$

With the aid of numerical method, we obtain the following Picard–Fuchs operator for a conjectural mirror family for $X_{\mathcal{P}_1}$,

$$\begin{aligned}\mathcal{D} = \theta^4 &- 2z(33\theta^4 + 58\theta^3 + 48\theta^2 + 19\theta + 3) \\ &+ 4z^2(174\theta^4 + 448\theta^3 + 527\theta^2 + 314\theta + 75) \\ &- 8z^3(332\theta^4 + 1096\theta^3 + 1507\theta^2 + 953\theta + 228) \\ &+ 96z^4(\theta+1)^2(6\theta+5)(6\theta+7),\end{aligned} \tag{52}$$

where $\theta = z\partial_z$ and $\mathcal{D}\omega_0(z) = 0$. We observe that the operator generates integral BPS numbers for genus 0 and genus 1 with small degrees, by standard methods for the computation [9, 6].

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