Mirror symmetry for complete intersection Calabi-Yau threefolds in Gorenstein minuscule Schubert varieties

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1 Minuscule Schubert varieties

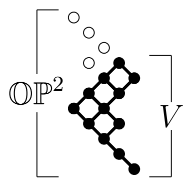
Let G be a simply-connected simple complex algebraic group, B a Borel subgroup and T a maximal torus such that $T \subset B \subset G$. We denote by W the associated Weyl group. We also fix a parabolic subgroup $P \supset B$. Let us denote by W_P the Weyl group of P and by W^P the set of minimal length representatives in W of the coset W/W_P. For any $w \in W^P$, We denote with $X(w) = \overline{BwP/P}$ the Schubert variety in G/P associated to w.

We assume that $P = P_\varpi$ a maximal parabolic subgroup with associated fundamental weight ϖ, and ϖ is minuscule.

$$\varpi \text{ is minuscule } \iff \langle \alpha^\vee, \varpi \rangle \leq 1 \text{ for all positive root } \alpha.$$

The minuscule homogeneous spaces G/P_ϖ are the Grassmannians, the quadrics, the orthogonal Grassmannians, and two exceptional varieties: the Cayley plane $\mathbb{OP}^2 = E_6/P_1$ and the Freudential variety E_7/P_7. The Schubert varieties in G/P_ϖ are then called minuscule Schubert varieties.

For the minuscule Schubert varieties $X(w)$, the Picard group is generated by a very ample invertible sheaf $\mathcal{L}_w$ and the basis of $H^0(X(w), \mathcal{L}_w)$ is parametrized by a distributive lattice $H_w \subset W^P$. Now we take $V = X(w)$ a particular Schubert variety in Cayley plane $\mathbb{OP}^2$ whose associated poset $D(H_w)$ is the following.



> **Theorem 1.1.** *There exists a smooth linear section Calabi-Yau threefold X in V. Its topological invariants are*
>
> $$\deg(X) = 33, \quad c_2(X) \cdot H = 78, \quad \chi(X) = -102.$$

Remark 1.2. This X is the essentially unique nontrivial example of smooth complete intersection Calabi-Yau threefolds in Gorenstein minuscule Schubert varieties. Another example is just a complete intersection in orthogonal Grassmannian $OG(5, 10)$.

2 Toric degenerations

Let $H = (H, \leq)$ be a finite distributive lattice and $D = D(H)$ the finite poset of nonzero join-irreducible elements of H. An element of H can be regarded as a lower subset in D.

$$H : \text{a finite distributive lattice} \xleftrightarrow{1:1} D : \text{a finite poset.}$$

Let us denote by S the set of *stars*, the additional maximal and minimal elements for D, by E the set of oriented edges in the Hasse diagram of $D \cup S$, and by $L(B)$ an $\mathbb{R}$-vector space generated by the basis B. We define a linear map $\lambda : L(H) \longrightarrow L(E)^*$ as

$$\lambda(\tau)(e) := \begin{cases} 1, & h(e) \in \tau \text{ and } t(e) \notin \tau, \\ 0, & \text{otewise,} \end{cases}$$

where $\tau \in H$ is regarded as a lower subset in $D \cup S$ and the head $h(e)$ and the tail $t(e) \in D \cup S$ are defined as follows:

$$t(e) \xrightarrow{e} h(e) \qquad \partial(e) = h(e) - t(e).$$

The polytope $\Delta_H := \mathrm{Conv}(\lambda(H)) \subset L(E)^*$ is actually defined in $M_\mathbb{R} := L(D)^* \subset L(E)^*$. The following lemma gives a combinatorial description of degenerations of Gonciulea-Lakshmibai.

Lemma 2.1. *Let $H = (H, \leq)$ be a distributive lattice, and assume that the k-algebra $R = k\left[p_\alpha \mid \alpha \in H\right]/I$ has the standard monomial basis $\{p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_r} (\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r)\}$ and the ideal I is generated by degree two staraightening relations:*

$$p_\tau p_\phi - p_{\tau\wedge\phi} p_{\tau\vee\phi} + \sum_{\substack{\alpha < \tau\wedge\phi \\ \tau\vee\phi < \beta}} c_{\alpha\beta} p_\alpha p_\beta$$

wehre τ, ϕ are non-comparable in H. Then Proj$R$ *degenerates to a normal toric variety $\mathbb{P}_{\Delta_H}$.*

Furthermore, if the poset $D \cup S$ has the constant maximal length c_1 of oriented paths, we can see the duality of reflexive polytopes:

$$(c_1 \cdot \Delta_H - (1, 1, \cdots, 1))^* \simeq \mathrm{Conv}(\delta(E))$$

where δ is the composition of boundary map ∂ and projection pr_1,

$$\delta : L(E) \xrightarrow{\partial} L(D) \oplus L(S)_\mathbb{R} \xrightarrow{\mathrm{pr}_1} L(D) =: N_\mathbb{R}.$$

> **Theorem 2.2.** *A Gorenstein minuscule Schubert variety $X(w)$ degenerates to the Gorenstein toric Fano variety $\mathbb{P}_{\Delta_{H_w}}$.*

Remark 2.3. It is also true for the Lagrangian Grassmannians with other distributive lattices.

3 Mirror symmetry for X

The mirror construction is based on the idea of conifold transitions. We present the power series expansion of the period map in terms of the dual graph of the Hasse diagram for $D_w \cup S$.

Proposition 3.1. *The conjectural mirror family $\mathcal{X}^\vee$ of X is defined over $\mathbb{P}^1$. The period integrals of this family satisfies the Picard-Fuchs equation $\mathcal{D}_x \omega(x) = 0$ with $\theta_x = x\frac{\mathrm{d}}{\mathrm{d}x}$ and*

$$\begin{aligned} \mathcal{D}_x = & 121\theta_x^4 - 77x(130\theta_x^4 + 266\theta_x^3 + 210\theta_x^2 + 77\theta_x + 11) \\ & - x^2(32126\theta_x^4 + 89990\theta_x^3 + 103725\theta_x^2 + 55253\theta_x + 11198) \\ & - x^3(28723\theta_x^4 + 74184\theta_x^3 + 63474\theta_x^2 + 20625\theta_x + 1716) \\ & - 7x^4(1135\theta_x^4 + 2336\theta_x^3 + 1881\theta_x^2 + 713\theta_x + 110) \\ & - 49x^5(\theta_x + 1)^4. \end{aligned}$$

Conjecture 3.2. *There is a Calabi-Yau threefold Y with Picard number one whose derived category is equivalent to that of X. Its topological invariants are*

$$\deg(Y) = 21, \quad c_2(Y) \cdot H = 66, \quad \chi(Y) = -102.$$

Remark 3.3. We observe the switching of BPS numbers.

d	$g = 0$ for X	$g = 1$ for X	$g = 0$ for Y	$g = 1$ for Y
1	252	0	387	0
2	1854	0	4671	0
3	27156	0	124323	1
4	567063	0	4782996	1854
5	14514039	4671	226411803	606294