Linear section Calabi–Yau threefolds in Hibi toric varieties

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1 Hibi toric varieties

Let $P = (P, \prec)$ be a finite poset. The order polytope $\Delta(P) \subset \mathbb{R}^{|P|}$ is defined as follows:

$$\Delta(P) := \left\{ x = (x_u)_{u \in P} \;\middle|\; 0 \leq x_u \leq x_v \leq 1 \text{ for all } u \prec v \in P \right\}.$$

The projective toric variety associated with $\Delta(P)$, i.e.

$$\mathbb{P}_{\Delta(P)} := \operatorname{Proj} \mathbb{C}[\operatorname{Cone}(\{1\} \times \Delta(P)) \cap (\mathbb{Z} \times \mathbb{Z}^{|P|})] \subset \mathbb{P}^{l(\Delta(P))-1}$$

is called the Hibi toric variety for P.

2 Simple posets

For posets P_1 and P_2, the sum $P_1 + P_2 := P_1 \sqcup P_2$ is the poset with the partial order $\prec$ extended from those on the posets P_1, P_2.

Lemma 2.1. $\mathbb{P}_{\Delta(P_1)} \times \mathbb{P}_{\Delta(P_2)} \simeq \mathbb{P}_{\Delta(P_1+P_2)}$.

The ordinary sum $P_1 \oplus P_2 := P_1 \sqcup P_2$ is the poset with the partial order $\prec$ extended from those on the posets P_1, P_2 and imposing $u \prec v$ for all $u \in P_1$ and $v \in P_2$.

Lemma 2.2. *1. A projective join of Hibi toric varieties $\mathbb{P}_{\Delta(P_1)}$, $\mathbb{P}_{\Delta(P_2)}$ in general $\mathbb{P}^{l(\Delta(P_1))-1}, \mathbb{P}^{l(\Delta(P_2))-1} \subset \mathbb{P}^{l(\Delta(P_1))+l(\Delta(P_2))-1}$ is isomorphic to the Hibi toric variety $\mathbb{P}_{\Delta(P_1 \oplus \{o\} \oplus P_2)}$.*

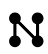
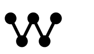
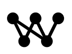
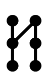
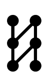
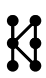
2. The Hibi toric variety $\mathbb{P}_{\Delta(P_1 \oplus P_2)}$ is isomorphic to a (special) hyperplane on $\mathbb{P}_{\Delta(P_1 \oplus \{o\} \oplus P_2)}$.

We call a poset P *simple* if it is neither $P_1 + P_2$ nor $P_1 \oplus P_2$ for non-empty posets P_1 and P_2.

3 Classification

We say that a finite poset P is *pure* if the length of maximal chains on P is a constant. For a pure poset P, we denote by h_P the length of maximal chains on $\widehat{P} := \left\{\widehat{0}\right\} \oplus P \oplus \left\{\widehat{1}\right\}$.

There are eight simple pure posets with $|P| - h_P \leq 2$ upto order duality, listed in the following table.

posets	•							
V	$\mathbb{P}^n$	$G(2,5)$	$LG(3,6)$				$G(2,6)$	$OG(5,10)$

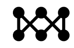
Each poset P defines a Gorenstein terminal Hibi toric variety with $-K_{\mathbb{P}_{\Delta(P)}} = \mathcal{O}(h_P)$. Some of them can be regarded as degeneration limits of linear sections of Fano varieties V with Picard number one.

Theorem 3.1. *There exist* 52 *distinct simple pure posets with $|P| - h_P = 3$ upto order duality. Each poset defines a family of linear section Calabi–Yau threefolds in the Hibi toric variety.*

Remark 3.2. These include the case of $V = G(2,7), G(3,6)$ and a Schubert variety $\mathbf{\Sigma} \subset \mathbb{OP}^2$.

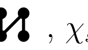
4 Calabi–Yau equations

We consider the diagonal subfamilies of the Batyrev–Borisov mirror families for linear section Calabi-Yau threefolds X in Hibi toric varieties. Some of them give us the fourth order differential operators which vanish the period integrals of the diagonal subfamilies.

Example 4.1 (new CYE)**.** In the case of  , $\chi_{st}(X) = -54$,

$$\begin{aligned}\mathcal{D}_x = \;&\theta^4 - 2x(3 + 19\theta + 48\theta^2 + 58\theta^3 + 33\theta^4) \\ &+ 4x^2(75 + 314\theta + 527\theta^2 + 448\theta^3 + 174\theta^4) \\ &- 8x^3(228 + 953\theta + 1507\theta^2 + 1096\theta^3 + 332\theta^4) \\ &+ 96x^4(1+\theta)^2(5+6\theta)(7+6\theta),\end{aligned}$$

where $\theta = x\frac{\mathrm{d}}{\mathrm{d}x}$.

Example 4.2 (two MUM points)**.** For  , $\chi_{st}(X) = -66$,

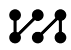
$$\begin{aligned}\mathcal{D}_x = \;&3721\theta^4 - 61x(305 + 1891\theta + 4677\theta^2 + 5572\theta^3 + 3029\theta^4) \\ &+ x^2(611586 + 2572675\theta + 4267228\theta^2 + 3428132\theta^3 + 1215215\theta^4) \\ &- 81x^3(37332 + 142191\theta + 206807\theta^2 + 140178\theta^3 + 39370\theta^4) \\ &+ 6561x^4(558 + 2241\theta + 3356\theta^2 + 2230\theta^3 + 566\theta^4) \\ &- 1594323x^5(1+\theta)^4.\end{aligned}$$

Conjecture 4.3. *If there exists the Calabi–Yau operator, the linear section Calabi–Yau threefolds in Hibi toric variety can be deformed into a smooth Calabi–Yau threefold.*

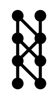
5 Non-simple posets

The Hadamard product of two differential equations with power series solutions around $x = 0$ given by $\sum_n A_n x^n$ and $\sum_n B_n x^n$ is the equation that has $\sum_n A_n B_n x^n$ as its power series solution.

Proposition 5.1. *If the Calabi–Yau operator exists for $P_1 \oplus P_2$, it becomes the Hadamard product of those for the posets P_1 and P_2 with the power series solutions corresponding to the monodromy invariant periods.*

Example 5.2 (direct sum)**.** In the case of  , $V = Q^3 \times Q^3$.

$$\begin{aligned}\mathcal{D}_x = \;&25\theta^4 - 20x(5 + 30\theta + 72\theta^2 + 84\theta^3 + 36\theta^4) \\ &- 16x^2(-35 - 70\theta + 71\theta^2 + 268\theta^3 + 181\theta^4) \\ &+ 256x^3(1+\theta)(165 + 375\theta + 248\theta^2 + 37\theta^3) \\ &+ 1024x^4(59 + 232\theta + 331\theta^2 + 198\theta^3 + 39\theta^4) \\ &+ 32768x^5(1+\theta)^4.\end{aligned}$$

Example 5.3 (projective join)**.** In the case of  , the linear section Calabi–Yau threefold can be deformed into a complete intersection of two Grassmannians, $G(2,5) \cap G(2,5) \subset \mathbb{P}^9$.