Hibi toric varieties and mirror symmetry

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Contents

Preface

In this thesis, we study the mirror symmetry for smooth Calabi–Yau 3-folds of Picard number one which degenerate to a general complete intersection in Hibi toric varieties. Two new examples of such Calabi–Yau 3-folds $\mathbf{\Sigma}(1^9)$ and $(G(2,5)^2)$ are our main interest because of its mirror symmetric property.

In Chapter 1, we collect the basic notations and results on Hibi toric varieties. Combinatorics of finite posets play an irreplaceable role for descriptions of the geometry of Hibi toric varieties.

In Chapter 2, we give a brief summary of the theory of toric degenerations. Especially, we study the Gonciulea–Lakshmibai degenerations [GL] from a viewpoint of our formulation of Hibi toric varieties.

In Chapter 3, we perform the conjectural mirror construction of smooth Calabi–Yau 3-folds of Picard number one which degenerate to complete intersections in Hibi toric varieties, based on the conjectural construction of [BCFKvS1] via conifold transition. We give an expression for the fundamental periods.

In Chapter 4, we study the examples of complete intersections in minuscule Schubert varieties. Listing all these Calabi–Yau 3-folds up to deformation equivalences, we find a new example $\mathbf{\Sigma}(1^9)$, a smooth complete intersection in a locally factorial Schubert variety $\mathbf{\Sigma}$ of the Cayley plane $\mathbb{OP}^2$. We calculate topological invariants of this Calabi–Yau 3-fold and conjecture that it has a non-trivial Fourier–Mukai partner.

In Chapter 5, we give an idea of regarding a complete intersection *of* projective varieties as a complete intersection of hyperplanes in the projective join of the varieties. We focus on an example $(G(2,5)^2)$, a complete intersection of two Grassmannians $G(2,5)$ with general positions in $\mathbb{P}^9$. We study the mirror symmetry for this Calabi–Yau 3-fold and suggest the possibility of a generalization of quantum hyperplane section theorem for subvarieties of high codimension.

In Appendix, we put the tables of BPS numbers computed using mirror symmetry.

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1

Hibi Toric Varieties

1.1 Order polytopes

Let $P = (P, \prec)$ be a finite partially ordered set (or *poset* for short) and $N = \mathbb{Z}P$, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ free abelian groups of rank $|P|$ dual to each other. We denote by $N_{\mathbb{R}}$, $M_{\mathbb{R}}$ the real scalar extensions $N \otimes_{\mathbb{Z}} \mathbb{R}$, $M \otimes_{\mathbb{Z}} \mathbb{R}$, respectively. We define the *order polytope* $\Delta(P) \subset M_{\mathbb{R}}$ as follows (cf. [Sta]).

$$\Delta(P) := \left\{ x = (x_u)_{u \in P} \,\middle|\, 0 \le x_u \le x_v \le 1 \text{ for all } u \prec v \in P \right\}. \tag{1.1.1}$$

It is easy to see that $\Delta(P)$ is an integral convex polytope of dimension $|P|$.

Definition 1.1.1. Let P be a finite poset and $\Delta(P)$ the order polytope for P. The projective toric variety associated with $\Delta(P)$,

$$\mathbb{P}_{\Delta(P)} := \mathrm{Proj}\, \mathbb{C}[\mathrm{Cone}(1 \times \Delta(P)) \cap (\mathbb{Z} \times M)] \tag{1.1.2}$$

is called the *Hibi toric variety* for a finite poset P.

Example 1.1.2. If P is a finite totally ordered set, the order polytope $\Delta(P)$ is a regular simplex of dimension $|P|$. Hence the corresponding Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is a projective space of dimension $|P|$.

Example 1.1.3. Assume that every pair of elements in a finite poset P is incomparable. In this case, the order polytope $\Delta(P)$ is $|P|$-dimensional hypercube $[0, 1]^{|P|}$. Then the corresponding Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is a $|P|$-times direct product of $\mathbb{P}^1$.

Example 1.1.4. One of the simplest examples of order polytopes is the Gelfand–Tsetlin polytopes for fundamental weights of special linear groups $SL(n + 1, \mathbb{C})$. The Gelfand–Tsetlin polytope for an integral dominant weight $\lambda = (\Lambda_0, \ldots, \Lambda_n) \in \mathbb{Z}^{n+1} / \langle (1, \ldots, 1) \rangle$ is defined by the following inequalities in $\mathbb{R}^{n(n+1)/2}$

$$\Lambda_n \leq x_{i+1,j+1} \leq x_{i,j} \leq x_{i,j+1} \leq \Lambda_i \quad \text{for all } 0 \leq i \leq j \leq n-1. \tag{1.1.3}$$

These inequalities can be represented in a diagram like as Figure 1.1.1 for $n = 3$.

$$\begin{array}{cccc} & & & \Lambda_0 \\ & & x_{02} & \\ & x_{01} & & \Lambda_1 \\ x_{00} & & x_{12} & \\ & x_{11} & & \Lambda_2 \\ & & x_{22} & \\ & & & \Lambda_3 \end{array}$$

Figure 1.1.1: the Gelfand–Tsetlin polytopes for $SL(4, \mathbb{C})$

The Gelfand–Tsetlin polytope for a fundamental weight $\lambda = (1, \ldots, 1, 0, \ldots, 0)$ is in fact the order polytope $\Delta(P)$ for a poset P, whose Hasse diagram has rectangle shape (cf. § 1.4). The corresponding Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is the toric variety $P(k, n+1)$ defined by [BCFKvS1].

1.2 Homogeneous coordinate rings

To introduce another description of Hibi toric varieties which is standard in literatures, we should prepare some further definitions. For a poset P, an *order ideal* is a subset $I \subset P$ with the property that

$$u \in I \text{ and } v \prec u \text{ imply } v \in I. \tag{1.2.1}$$

A *lattice* L is a poset for which each pair of elements $\alpha, \beta \in L$ has the least upper bound $\alpha \vee \beta$ (called the join) and the greatest lower bound $\alpha \wedge \beta$ (called the meet) in L. A *distributive lattice* is a lattice on which the following identity holds for all triple elements $\alpha, \beta, \gamma \in L$,

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma). \tag{1.2.2}$$

For a finite poset P, the order ideals of P form a distributive lattice $J(P)$ with the partial order given by set inclusions. The join and the meet on $J(P)$ correspond to the set union and the set intersection, respectively. An example of a finite poset P and the distributive lattice $J(P)$ is depicted in Figure 1.2.1 using the Hasse diagram of posets (cf. § 1.4).

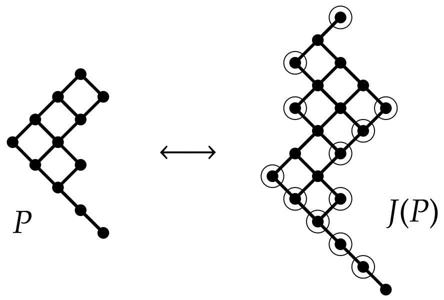


Figure 1.2.1: Hasse diagrams of P and $J(P)$

Let $J(P)$ be the distributive lattice of order ideals of a finite poset P. Denote by $\mathbb{C}[J(P)]$ the polynomial ring over $\mathbb{C}$ in $|J(P)|$ indeterminates p_α ($\alpha \in J(P)$). Let $I(J(P)) \subset \mathbb{C}[J(P)]$ the homogeneous ideal generated by the following binomial relations:

$$p_\tau p_\phi - p_{\tau\wedge\phi} p_{\tau\vee\phi} \ (\tau \nsim \phi), \tag{1.2.3}$$

where $\tau \nsim \phi$ denotes the pair of elements $\tau, \phi \in J(P)$ incomparable. One can check that the graded algebra $A_{J(P)} := \mathbb{C}[J(P)] / I(J(P))$ with the standard $\mathbb{N}$-grading inherited from $\mathbb{C}[J(P)]$ coincides with the homogeneous coordinate ring of the Hibi toric variety $\mathbb{P}_{\Delta(P)}$ with the embedding defined by the very ample line bundle associated with the order polytope $\Delta(P)$. The graded algebra $A_{J(P)}$ is usually called the *Hibi algebra* on the distributive lattice $J(P)$ (cf. [Hib]).

Remark 1.2.1. One may define the Hibi algebra A_L for not only $J(P)$ but also any finite distributive lattice L. In fact, it does not make differences because of the Birkhoff representation theorem in the following. Let L be a finite lattice. It is easy to see that L has the unique maximal and minimal element with respect to the partial order on L. An element $\alpha \in L$ is said to be *join irreducible* if α is neither the minimal element nor the join of a finite set of other elements.

Theorem 1.2.2 (cf. [Bir])**.** *Let P be a finite poset and $J(P)$ the distributive lattice of order ideals of P. The full subposet of join irreducible elements of $J(P)$ coincides with P as a poset. This gives a one-to-one correspondence between finite posets and finite distributive lattices.*

As an example of this correspondence, a circled vertex of $J(P)$ in Figure 1.2.1 represents a join irreducible element of $J(P)$, i.e., the vertex with exactly one edge below. We can easily reconstruct the poset P as the set of circled vertices with the induced order in $J(P)$.

Example 1.2.3. Let P be a finite poset and $P^* = P \cup \{\hat{1}\}$ the poset with extended partial order on P with $u \prec \hat{1}$ for all $u \in P$. The Hibi toric variety $\mathbb{P}_{\Delta(P^*)}$ is a projective cone over

$\mathbb{P}_{\Delta(P)}$ in $\mathbb{P}^{|J(P)|}$. In fact, the variable $p_{\hat{1}}$ in the homogeneous coordinate ring $A_{J(P^*)}$ does not involve in any relation $p_\tau p_\phi - p_{\tau\wedge\phi}p_{\tau\vee\phi}$ ($\tau \nsim \phi$).

1.3 Projective joins

We give a generalization of the projective cone given in Example 1.2.3. First, let us recall the definition of projective joins of projective varieties.

Definition 1.3.1. Let $V_1 \subset \mathbb{P}^n_1$ and $V_2 \subset \mathbb{P}^m_2$ be projective varieties in projective subspaces $\mathbb{P}^n_1, \mathbb{P}^m_2 \subset \mathbb{P}^{n+m+1}$ with general positions. The projective join $J(V_1, V_2)$ of V_1 and V_2 is the union of all projective lines in $\mathbb{P}^{n+m+1}$ passing through a point of V_1 and a point of V_2.

It is natural to introduce the combinatorial analogue of this notion.

Definition 1.3.2. (1) Let Δ_1 and Δ_2 be integral convex polytopes in $M^1_{\mathbb{R}}$ and $M^2_{\mathbb{R}}$, respectively. The *projective join* $J(\Delta_1, \Delta_2)$ of Δ_1 and Δ_2 is the convex hull of the sets $(0, \Delta_1, \mathbf{0})$ and $(1, \mathbf{0}, \Delta_2)$ in $\mathbb{R} \oplus M^1_{\mathbb{R}} \oplus M^2_{\mathbb{R}}$, where $\mathbf{0} \in M^j_{\mathbb{R}}$ is the origin for $j = 1, 2$.

(2) Let P_1 and P_2 be finite posets. The *projective join* $J(P_1, P_2)$ of P_1 and P_2 is the poset $P_1 \cup P_2 \cup \{o\}$ with the partial order $\prec$ extended from those on P_1 and P_2 by adding $u \prec o \prec v$ for all $u \in P_1$ and $v \in P_2$.

An example of the projective join $J(P, P)$ of finite posets P is depicted in Figure 1.3.1, again using the Hasse diagram of posets. The Hasse diagram of P is shaped like a rectangle and the middle vertex corresponds to the additional element o. Note that the definition of the projective join $J(P_1, P_2)$ is not symmetric for P_1 and P_2.

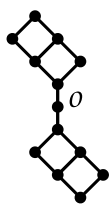


Figure 1.3.1: Projective join $J(P, P)$

Lemma 1.3.3. *Under the notations in Definition 1.3.1, it holds that*

(1) $\mathbb{P}_{J(\Delta_1,\Delta_2)} \simeq J(\mathbb{P}_{\Delta_1}, \mathbb{P}_{\Delta_2})$,

(2) $\Delta(J(P_1, P_2)) \simeq J(\Delta(P_1), \Delta(P_2))$.

Proof. (1) It is easy to see that the homogeneous coordinate ring of $J(\mathbb{P}_{\Delta_1}, \mathbb{P}_{\Delta_2})$ is isomorphic to that of $\mathbb{P}_{J(\Delta_1,\Delta_2)}$ by considering the polynomial relations in two kinds of variables corresponding to the coordinates on projective subspaces $\mathbb{P}^n_1$ and $\mathbb{P}^m_2$.

(2) The claim follows from the fact that $\Delta(J(P_1, P_2))$ is a convex hull of sets $(0, \Delta(P_1), \mathbf{0})$ and $(1, \mathbf{1}, \Delta(P_2))$, where $\mathbf{1} \in M^1_{\mathbb{R}}$ is the point with all $x_u = 1$ and $\mathbf{0} \in M^2_{\mathbb{R}}$ is the origin. In fact, every point $(x_o, (x_u), (x_v)) \in \Delta(J(P_1, P_2))$ is contained in a segment of the end points $(0, (\frac{x_u - x_o}{1 - x_o}), \mathbf{0})$ and $(1, \mathbf{1}, (\frac{x_v}{x_o})) \in \Delta(J(P_1, P_2))$.

□

Remark 1.3.4. Although the definition of $J(P_1, P_2)$ is not symmetric for P_1 and P_2, the order polytope $\Delta(J(P_1, P_2))$ is defined in a symmetric way up to unimodular transformations as we see from Lemma 1.3.3 (2) or the proof of it.

Corollary 1.3.5. *Let P_1 and P_2 be finite posets. The Hibi toric variety $\mathbb{P}_{\Delta(J(P_1,P_2))}$ is isomorphic to the projective join $J(\mathbb{P}_{\Delta(P_1)}, \mathbb{P}_{\Delta(P_2)})$ of Hibi toric varieties $\mathbb{P}_{\Delta(P_1)}$, $\mathbb{P}_{\Delta(P_2)}$.*

1.4 Invariant subvarieties

A nice property of Hibi toric varieties is that torus invariant subvarieties in Hibi toric varieties are also Hibi toric varieties. Before we see this, let us introduce some further combinatorial definitions.

For a finite poset P and a pair of elements $u, v \in P$, we say that *u covers v* if $u > v$ and there is no $w \in P$ with $u > w > v$. The *Hasse diagram* of a poset P is the oriented graph with vertex set P, having an edge $e = \{u, v\}$ going down from u to v whenever u covers v in P. Denote that the source $s(e) = u$ and the target $t(e) = v$ for an edge $e = \{u, v\}$ of the Hasse diagram of P if u covers v. Let us define the poset $\hat{P} := P \cup \left\{\hat{0}, \hat{1}\right\}$ by extending the partial order on P with $\hat{0} < u < \hat{1}$.

The defining inequalities of an order polytope $\Delta(P)$ are generated by $x_{s(e)} \geq x_{t(e)}$ for all $e \in E$, where E is the set of edges of the Hasse diagram of $\hat{P} = P \cup \left\{\hat{0}, \hat{1}\right\}$ and $x_{\hat{0}} = 0$ and $x_{\hat{1}} = 1$. We can get a face of $\Delta(P)$ by replacing some of these inequalities with equalities as we see below. Recall that a *full subposet* $y \subset \hat{P}$ is a subset of $\hat{P}$ whose poset structure is that inherited from $\hat{P}$. We call a full subposet $y \subset \hat{P}$ *connected* if all the elements in y are connected by edges in the Hasse diagram of y, and *convex* if $u, v \in y$ and $u < w < v$ imply $w \in y$.

Definition 1.4.1. Let P be a finite poset and $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$. A surjective map f from $\hat{P}$ to a finite set $\hat{P}' = P \cup \{\hat{0}, \hat{1}\}$ is called a *contraction* of $\hat{P}$ if every fiber $f^{-1}(i)$ $(i \in \hat{P}')$ is connected full subposet of $\hat{P}$ not containing both $\hat{0}$ and $\hat{1}$, and the following condition holds for all $u_k, v_k \in f^{-1}(k)$ and $i \neq j \in \hat{P}'$:

$$\text{a relation } u_i \prec u_j \text{ implies } v_i \nsucc v_j.$$

Remark 1.4.2. A contraction $f : \hat{P} \to \hat{P}'$ gives a natural partial order on the image set $\hat{P}'$, i.e. the partial order generated by the following relations:

$$i \prec j \Leftrightarrow \text{there exist } u \in f^{-1}(i) \text{ and } v \in f^{-1}(j) \text{ such that } u \prec v \text{ in } \hat{P}.$$

Further, $\hat{P}'$ turns out to be a so-called bounded poset by setting $\hat{1} \in f^{-1}(\hat{1})$ and $\hat{0} \in f^{-1}(\hat{0})$. Hence in fact, the above definition of contraction coincides with the more abstract definition in [Wag]; the fiber-connected tight surjective morphism of bounded posets.

For a contraction $f : \hat{P} \to \hat{P}'$, the corresponding face of $\Delta(P)$ is given by

$$\theta_f := \left\{ x \in \Delta(P) \;\middle|\; x_u = x_v \text{ for all } u, v \in f^{-1}(i) \text{ and } i \in \hat{P}' \right\}. \tag{1.4.1}$$

Conversely, we can reconstruct the contraction from each face $\theta_f \subset \Delta(P)$ by looking at the coordinates of general point in θ_f. Now we can rephrase the classical fact on the face structure of order polytopes in our terminology (cf. [Wag, Theorem 1.2]).

Proposition 1.4.3. *Let P be a finite poset, and $\Delta(P)$ the associated order polytope. The above construction gives a one-to-one correspondence between the faces of $\Delta(P)$ and the contractions of $\hat{P}$. Moreover, an inclusion of the faces corresponds to a composition of contractions.*

Remark 1.4.4. It is obvious that the face $\theta_{\hat{P} \to \hat{P}'} \subset \Delta(P)$ coincides with the $|P'|$-dimensional order polytope $\Delta(P')$ under a suitable choice of subspace of $M_{\mathbb{R}}$ and a unimodular transformation. This means that the torus invariant subvarieties in Hibi toric varieties are also Hibi toric varieties as noted before.

Finally, we note on divisors on a Hibi toric variety $\mathbb{P}_{\Delta(P)}$. Weil and Cartier divisors are naturally described in terms of the poset P. In fact, prime invariant Weil divisors correspond to the set of edges E of the Hasse diagram of $\hat{P}$ from Proposition 1.4.3. It is elementary to show that the divisor class group $\mathrm{Cl}(\mathbb{P}_{\Delta(P)})$ is a free $\mathbb{Z}$-module of rank $|E| - |P|$ and the Picard group $\mathrm{Pic}(\mathbb{P}_{\Delta(P)})$ is a free $\mathbb{Z}$-module whose rank coincides with the number of connected components of the Hasse diagram of P.

1.5 Singular loci

Let P be a finite poset. A *chain* of length k in P is a sequence of elements $u_0 \prec u_1 \prec \cdots \prec u_k \in P$ for $1 \le i \le k$. A chain is called *maximal* if there is no $v \prec u_0$ or $w \succ u_k$ in P and u_i covers u_{i-1} for all $1 \le i \le k$. We call a finite poset P is *pure* if every maximal chain has the same length. We have some known useful results on singularities of Hibi toric varieties.

Proposition 1.5.1 ([HH, Remark 1.6 and Lemma 1.4])**.** *Let P be a finite poset. The Hibi algebra $A_{J(P)}$ is Gorenstein if and only if P is pure. In this case the Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is a Gorenstein Fano variety with at worst terminal singularities.*

Theorem 1.5.2 ([Wag, Theorem 2.3 and Proof of Corollary 2.4])**.** *Let $\mathbb{P}_{\Delta(P)}$ be a Hibi toric variety for a finite poset P. A face $\theta_f \subset \Delta(P)$ corresponds to an irreducible component of the singular loci of $\mathbb{P}_{\Delta(P)}$ if and only if one of the fibers $f^{-1}(i)$ of the contraction $f : \hat{P} \to \hat{P}'$ is a minimal convex cycle in the Hasse diagram of $\hat{P}$ and all other fibers $f^{-1}(j)$ $(j \neq i)$ consist of one element.*

1.6 Gorenstein Hibi toric varieties

We give some further preliminary results for Gorenstein Hibi toric varieties, which are particularly important for our purpose. A finite poset P has a height function h by defining $h(u)$ to be the length of the longest chain bounded above by u in $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$. We define the *height* h_P of P as $h(\hat{1})$. For example, $h_P = 9$ for the pure poset P in Figure 1.2.1.

Suppose that P is pure and $J(P)$ the associated distributive lattice of order ideals of P. The associated Hibi toric variety $\mathbb{P}_{\Delta(P)} \subset \operatorname{Proj} \mathbb{C}[J(P)]$ is Gorenstein (Proposition 1.5.1) with the anticanonical sheaf $-\mathcal{K}_{\mathbb{P}_{\Delta(P)}} = \mathcal{O}(h_P)$. In fact, for prime invariant Weil divisors $D_e (e \in E)$ on $\mathbb{P}_{\Delta(P)}$, a linear equivalence is generated by the relations

$$\sum_{s(e)=u} D_e \simeq \sum_{t(e)=u} D_e, \tag{1.6.1}$$

for each $u \in P$ and $\mathcal{O}(D_{E^k})$ coincides with $\mathcal{O}(1)$ for each $E^k := \{e \in E \mid h(s(e)) = k\}$ and $D_{E^k} := \sum_{e \in E^k} D_e$ for $k = 1, \ldots, h_P$.

Let us define $\Delta := \sum_{u \in P} h(u)\chi_u - h_P \Delta(P)$, a polytope corresponding to the anticanonical sheaf $-\mathcal{K}_{\mathbb{P}_{\Delta(P)}} = \mathcal{O}(D_E) = \mathcal{O}(h_P)$ containing the origin $\mathbf{0} \in M_{\mathbb{R}}$ as an internal integral point. Since $\mathbb{P}_{\Delta(P)}$ is Gorenstein, Δ turns out to be a *reflexive polytope*, i.e., it contains the unique internal integral point $\mathbf{0}$ and every facet has integral distance one to $\mathbf{0}$ [Bat1, Theorem 4.1.9]. We remark that the polar dual polytope $\Delta^* \subset N_{\mathbb{R}}$ of Δ also has a good description [BCFKvS2] [HH]. The abelian groups $\mathbb{Z}\hat{P} = N \oplus \mathbb{Z}\left\{\hat{0}, \hat{1}\right\}$ and $\mathbb{Z}E$, respectively, may be viewed as the groups of 0-chains and 1-chains of the natural chain complex associated with the Hasse diagram of $\hat{P}$. The boundary map in the chain complex is

$$\partial : \mathbb{Z}E \longrightarrow \mathbb{Z}\hat{P}, \quad e \mapsto t(e) - s(e). \tag{1.6.2}$$

We also consider the projection $\mathrm{pr}_1 : \mathbb{Z}\hat{P} \to N$ and the composed map

$$\delta := \mathrm{pr}_1 \circ \partial : \mathbb{Z}E \longrightarrow N. \tag{1.6.3}$$

The dual polytope Δ^* coincides with the convex hull of the image $\delta(E) \subset N_{\mathbb{R}}$. Further, the linear map δ gives a bijection between E and the set of vertices in Δ^*.

2
Toric Degenerations

2.1 Generalities on toric degenerations

We follows the formulation by [CHV] and [And] [Kav]. Let A be a $\mathbb{C}$-algebra and $(\mathbb{Z}^n, <)$ a totally ordered group, i.e., $<$ is a total order on a free abelian group $\mathbb{Z}^n$ such that $a < b$ implies $a + c < b + c$ for all $a, b, c \in \mathbb{Z}^n$. A $\mathbb{Z}^n$*-filtration* $\mathcal{F}$ on A is a family of $\mathbb{C}$-subspaces $\mathcal{F}_a A \subset A$ ($a \in \mathbb{Z}^n$) satisfying the following four conditions:

(1) $\mathcal{F}_a A \subset \mathcal{F}_b A$ (for all $a < b$),

(2) $\bigcup_{a \in \mathbb{Z}^n} \mathcal{F}_a A = A$,

(3) $(\mathcal{F}_a A)(\mathcal{F}_b A) \subset \mathcal{F}_{a+b} A$ (for all $a, b \in \mathbb{Z}^n$) and

(4) $1 \in \mathcal{F}_0 A \setminus \mathcal{F}_{<0} A$.

Suppose that $A = \bigoplus_{k=0}^{\infty} A_k$ is a graded $\mathbb{C}$-algebra with $A_0 = \mathbb{C}$ and $\dim A_k < \infty$ for all $k \in \mathbb{N}$. A *graded* $\mathbb{Z}^n$*-filtration* on A is a $\mathbb{Z}^n$-filtration compatible with the grading, i.e., $\mathcal{F}_a A \cap A_k \neq \emptyset \Rightarrow A_l \subset \mathcal{F}_a A$ for all $l < k$. Denote by the same symbol $\mathcal{F}$ the $(\mathbb{Z} \times \mathbb{Z}^n)$-filtration on A defined as $\mathcal{F}_{(k,a)} A := \mathcal{F}_a A \cap \oplus \bigoplus_{l \leq k} A_l$ with the total order $<$ on $\mathbb{Z} \times \mathbb{Z}^n$ lexicographically extended from that on $\mathbb{Z}^n$. For any nonzero $f \in A$, there is the smallest $a \in \mathbb{Z}^n$ (called the *order* of f and denoted by $\operatorname{ord}_{\mathcal{F}} f$) such that $f \in \mathcal{F}_a A$. It holds that $\mathbf{0} \leq \operatorname{ord}_{\mathcal{F}} f \in \mathbb{N} \times \mathbb{Z}^n$ for all $0 \neq f \in A$. We may define the *associated* $(\mathbb{N} \times \mathbb{Z}^n)$*-graded algebra* of A as

$$\operatorname{gr}_{\mathcal{F}} A = \bigoplus_{(k,a) \in \mathbb{N} \times \mathbb{Z}^n} \mathcal{F}_{(k,a)} A / \mathcal{F}_{<(k,a)} A. \tag{2.1.1}$$

As [Cal, § 3.2] [AB, Proposition 2.2] [And, Proposition 5.1], one can prove the following.

Proposition 2.1.1. *Let $(\mathbb{Z}^n, <)$ be a totally ordered group, A a graded $\mathbb{C}$-algebra with $A_0 = \mathbb{C}$ and $\dim A_k < \infty$ for all $k \in \mathbb{N}$ and $\mathcal{F}$ a graded $\mathbb{Z}^n$-filtration on A. Assume that $\operatorname{gr}_{\mathcal{F}} A$ is finitely generated. Then there is a finitely generated flat graded $\mathbb{C}[t]$-algebra $\mathcal{A} \subset A[t]$ such that*

(1) $\mathcal{A}/t\mathcal{A} \simeq \operatorname{gr}_{\mathcal{F}} A$, and

(2) $\mathcal{A}[t^{-1}] \simeq A[t, t^{-1}]$ as $\mathbb{C}[t, t^{-1}]$-algebras.

Geometrically, Proposition 2.1.1 says there is a projective flat family $\operatorname{Proj} \mathcal{A} \to \mathbb{C}$ with general fiber isomorphic to $\operatorname{Proj} A$ and special fiber $\operatorname{Proj}(\operatorname{gr}_{\mathcal{F}} A)$.

Corollary 2.1.2 (Toric degeneration)**.** *Let $(\mathbb{Z}^n, <)$ be a totally ordered group and A a graded $\mathbb{C}$-algebra with a graded $\mathbb{Z}^n$-filtration $\mathcal{F}$ with one-dimensional leaves, i.e., for all $(k, a) \in \mathbb{N} \times \mathbb{Z}^n$, $\dim \mathcal{F}_{(k,a)}A/\mathcal{F}_{<(k,a)}A \leq 1$. Assume that $\operatorname{gr}_{\mathcal{F}} A$ is a finitely generated integral domain. Then $\operatorname{gr}_{\mathcal{F}} A$ is a semigroup ring $\mathbb{C}[\Gamma]$ associated with the semigroup $\Gamma := \left\{(k, a) \mid \dim \mathcal{F}_{(k,a)}A/\mathcal{F}_{<(k,a)}A = 1\right\} \subset \mathbb{N} \times \mathbb{Z}^n$ and the projective variety $\operatorname{Proj} A$ degenerates to the projective toric variety $\operatorname{Proj}(\operatorname{gr}_{\mathcal{F}} A)$.*

2.2 Standard monomial basis

Let A be a graded $\mathbb{C}$-algebra and $\mathbb{C}[p]$ be a polynomial ring in n indeterminates p_j ($j = 1, \dots, n$) with standard grading. Assume that there exists a surjective homomorphism $\phi : \mathbb{C}[p] \to A$ as graded $\mathbb{C}$-algebras, i.e., $A \simeq \mathbb{C}[p]/I$ where $I := \ker \phi$ a homogeneous ideal. A $\mathbb{C}$-basis of A represented as a certain set of monomials in $\mathbb{C}[p]$ is called a *standard monomial basis* of A if it exists.

Example 2.2.1. Let $\prec$ be a term order on $\mathbb{C}[p]$ and denote by $\operatorname{in}_{\prec} f$ the initial term of $f \in \mathbb{C}[p]$ with respect to $\prec$. The *initial ideal* $\operatorname{in}_{\prec} I$ of an ideal $I \subset \mathbb{C}[p]$ is defined as a $\mathbb{C}$-space $\operatorname{in}_{\prec} I := \mathbb{C}\{\operatorname{in}_{\prec} f \mid f \in I\}$. Then the set $\{p^m \notin \operatorname{in}_{\prec} I\}$ is called a standard monomial basis of $A = \mathbb{C}[p]/I$ with respect to $\prec$. In fact, it is a $\mathbb{C}$-basis of A because every nonzero polynomial $r(p) \in I$ includes a term in $\operatorname{in}_{\prec} I$ and the reduction algorithm works.

Example 2.2.2. Let P be a finite poset, $J(P)$ the distributive lattice of order ideals of P and A a graded $\mathbb{C}$-algebra. Assume that there exists a surjective homomorphism $\phi : \mathbb{C}[J(P)] \to A$ as graded $\mathbb{C}$-algebras, where $\mathbb{C}[J(P)]$ is a polynomial ring in $|J(P)|$ indeterminates p_α ($\alpha \in J(P)$). Then the set $\{p_{\tau_1} p_{\tau_2} \cdots p_{\tau_r} + I \mid \tau_1 \preceq \tau_2 \preceq \cdots \preceq \tau_r\}$ is called standard monomial basis of A with respect to $J(P)$ if it is a $\mathbb{C}$-basis of A.

2.3 Gonciulea–Lakshmibai degenerations

We recall the result of Gonciulea and Lakshmibai [GL] as an important example of toric degenerations. We give a simple proof of this theorem in our terminology.

Theorem 2.3.1 ([GL]). *Let P be a finite poset and $A \simeq \mathbb{C}[J(P)]/I$ a graded $\mathbb{C}$-algebra which has a standard monomial basis with respect to $J(P)$ in the sense of Example 2.2.2. Assume that the homogeneous ideal I is generated by the following relations:*

$$p_\tau p_\phi - p_{\tau\wedge\phi} p_{\tau\vee\phi} + \sum_{\substack{\alpha<\tau\wedge\phi \\ \tau\vee\phi<\beta}} c_{\alpha\beta} p_\alpha p_\beta \tag{2.3.1}$$

for all $\tau \nsim \phi$. Then the variety Proj A *degenerates to the Hibi toric variety $\mathbb{P}_{\Delta(P)}$.*

Proof. We construct a $M \simeq \mathbb{Z}^{|P|}$-filtration $\mathcal{F}$ on A by setting:

$$\operatorname{ord}_{\mathcal{F}}(p_{\tau_1} p_{\tau_2} \cdots p_{\tau_r} + I) := \sum_{i=1}^{r} \chi(\tau_i) \quad (\text{for all } \tau_1 \leq \tau_2 \leq \cdots \leq \tau_r \in J(P)), \tag{2.3.2}$$

where $\chi(\tau) := \sum_{v\in\tau} (\delta_{uv})_{u\in P}$ and we take a reverse lexicographic order on M for a linear extension of the partial order $<$ on P. We verify the four axioms of filtration and that it becomes a graded M-filtration. The value of $\operatorname{ord}_{\mathcal{F}}$ are all different for distinct standard monomials because we can always recover all $\tau_i = \{d \in P \mid \operatorname{ord}_{\mathcal{F}}(p_{\tau_1} p_{\tau_2} \cdots p_{\tau_r} + I)(d) \leq i\}$ from that value. In addition we can check $\operatorname{ord}_{\mathcal{F}}(p_\alpha p_\beta) < \operatorname{ord}_{\mathcal{F}}(p_{\tau\wedge\phi} p_{\tau\vee\phi})$ for all $\alpha < \tau \wedge \phi$ and $\tau \vee \phi < \beta$ directly from the definition of the order on M. Then we conclude $\operatorname{gr}_{\mathcal{F}} A \simeq A_{J(P)}$ and the claim from Corollary 2.1.2. □

Many studies on the standard monomial theory for flag varieties and Schubert varieties often give examples which can be applied Theorem 2.3.1. For instance, the standard monomial theory for the so-called minuscule Schubert varieties [LMS] gives an examples. The terminology in the following theorem will be introduced in § 4.

Theorem 2.3.2 ([GL]). *A minuscule Schubert variety $X(w)$ degenerates to the Hibi toric variety $\mathbb{P}_{\Delta(P_w)}$, where P_w is the minuscule poset for $X(w)$.*

We give a corollary in another direction.

Corollary 2.3.3. *Let P_1, P_2 be finite posets and $A^1 \simeq \mathbb{C}[J(P_1)]/I_1, A^2 \simeq \mathbb{C}[J(P_2)]/I_2$ be $\mathbb{C}$-graded algebras satisfying the assumptions in Theorem 2.3.1, respectively. Then the projective join $J(\operatorname{Proj} A_1, \operatorname{Proj} A_2)$ degenerates to the Hibi toric variety $\mathbb{P}_{\Delta(J(P_1,P_2))} \simeq J(\mathbb{P}_{\Delta(P_1)}, \mathbb{P}_{\Delta(P_2)})$ (cf. Corollary 1.3.5).*

Proof. It is easy to check that the homogeneous coordinate ring $A \simeq \mathbb{C}[J(P_1, P_2)]/I$ of $J(\operatorname{Proj} A_1, \operatorname{Proj} A_2)$ satisfies all assumptions in Theorem 2.3.1 because the ideal $I \subset \mathbb{C}[J(P_1, P_2)]$ is generated by generators of the ideals $I_1 \subset \mathbb{C}[J(P_1)]$ and $I_2 \subset \mathbb{C}[J(P_2)]$ regarded as elements in $\mathbb{C}[J(P_1, P_2)]$. □

3

Mirror Symmetry

3.1 Batyrev–Borisov construction

First we apply the Batyrev–Borisov mirror construction [Bat1] [Bor] to Calabi–Yau complete intersections in Gorenstein Hibi toric varieties. Let P be a finite pure poset and $N = \mathbb{Z}P$ and $M = \mathrm{Hom}(N, \mathbb{Z})$ dual free abelian groups as before. The order polytope $\Delta(P) \subset M_{\mathbb{R}}$ is a $|P|$-dimensional integral polytope associated with the hyperplane class on the Gorenstein Hibi toric variety $\mathbb{P}_{\Delta(P)}$. We use the same notations as in § 1.6, a reflexive polytope $\Delta = \sum_{u \in P} h(u)\chi_u - h_P\Delta(P) \subset M_{\mathbb{R}}$, the polar dual polytope $\Delta^* = \mathrm{Conv}\,\delta(E) \subset N_{\mathbb{R}}$, Weil divisors $D_{E'} = \sum_{e \in E'} D_e$ $(E' \subset E)$ and so on.

Let $X_0 \subset \mathbb{P}_{\Delta(P)}$ be a general Calabi–Yau complete intersection of degree $(d_1, \ldots, d_r)$ with respect to $\mathcal{O}(1)$. That is, $d_1, \ldots, d_r$ satisfies $\sum_{j=1}^{r} d_j = h_P$. We choose a *nef-partition* of Δ, a special kind of Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ of Δ, in the following specific way. Define subsets E_j of edges in E as

$$E_j = \bigcup_{k=d_1+\cdots+d_{j-1}+1}^{d_1+\cdots+d_j} E^k, \tag{3.1.1}$$

where $E^k = \{e \in E \mid h(s(e)) = k\}$. It turns out that $\mathcal{O}(D_{E_j}) = \mathcal{O}(d_j)$ and a nef-partition is obtained from $E = E_1 \cup E_2 \cup \cdots \cup E_r$. Define $\nabla_j = \mathrm{Conv}(\{0\}, \delta(E_j))$ and the Minkowski sum $\nabla = \nabla_1 + \cdots + \nabla_r \subset N_{\mathbb{R}}$. From [Bor], it holds that

$$\Delta^* = \mathrm{Conv}(\nabla_1, \ldots, \nabla_r), \quad \nabla^* = \mathrm{Conv}(\Delta_1, \ldots, \Delta_r) \quad \text{and} \quad \Delta = \Delta_1 + \cdots + \Delta_r, \tag{3.1.2}$$

where Δ_j is the integral polytope in $M_{\mathbb{R}}$ defined by $\left\langle \Delta_i, \nabla_j \right\rangle \geq -\delta_{ij}$. The explicit expressions of Δ_j and ∇^* are as follows:

$$\Delta_j = d_j\mathbf{1} - \sum_{i=1}^{d_j} \chi_{d_1+\cdots+d_{j-1}+i} - d_j\Delta(P) = \mathrm{Conv}\left\{ d_j\chi(\tau) - \sum_{i=1}^{d_j} \chi_{d_1+\cdots+d_{j-1}+i} \;\middle|\; \tau \in J(P) \right\}, \quad (3.1.3)$$

$$\nabla^* = \mathrm{Conv}\left\{ d_j\chi(\tau) - \sum_{i=1}^{d_j} \chi_{d_1+\cdots+d_{j-1}+i} \;\middle|\; \tau \in J(P), 1 \leq j \leq r \right\}, \quad (3.1.4)$$

where $\chi_j := \chi(\tau_j) = \sum_{v\in\tau_j}(\delta_{uv})_{u\in P}$ with $\tau_j := \{u \in P \mid h(u) < j\}$.

Now we introduce the Batyrev–Borisov mirror of $Y = \hat{X}_0$, the strict transform of X_0 in a MPCP-resolution $\hat{\mathbb{P}}_{\Delta(P)}$ of $\mathbb{P}_{\Delta(P)}$ defined by [Bat1]. The mirror of $Y \subset \hat{\mathbb{P}}_{\Delta(P)}$ is birational to the set given by the following equations in torus $(\mathbb{C}^*)^{|P|}$:

$$\tilde{f}_j = 1 - (\sum_{e\in E_j} a_e t^{\delta(e)}) = 0 \quad \text{(for all } 1 \leq j \leq r), \quad (3.1.5)$$

where each $a_e \in \mathbb{C}$ is a parameter. Further, the precise mirror Calabi–Yau variety Y^* of Y is obtained as the closure of the above set in MPCP-resolution $\hat{\mathbb{P}}_\nabla$ of $\mathbb{P}_\nabla$.

The mirror $Y^* \subset \hat{\mathbb{P}}_\nabla$ actually has the expected stringy (or string-theoretic) Hodge numbers as proved in [BB1, BB2] and is smooth in 3-dimensional case. The stringy Hodge numbers of X_0 coincide with the usual Hodge numbers of Y if there exists a crepant resolution $Y \to X_0$. Applying their formula for stringy $(1,*)$-Hodge numbers to the case of Calabi–Yau complete intersections X_0 in Gorenstein Hibi toric varieties $\mathbb{P}_{\Delta(P)}$, we obtain the following convenient expressions in terms of the poset P.

Theorem 3.1.1 (cf. [BB1, Proposition 8.6])**.** *The stringy $(1,*)$-Hodge numbers of a general Calabi–Yau complete intersections X_0 of degree $(d_1,\ldots,d_r)$ in a Gorenstein Hibi toric variety $\mathbb{P}_{\Delta(P)}$ are given by the following formulae*

$$\begin{gathered} h^{1,1}_{\mathrm{st}}(X_0) = |E| - |P|, \qquad h^{1,k}_{\mathrm{st}}(X_0) = 0 \quad (1 < k < |P| - r - 1), \\ h^{1,|P|-r-1}_{\mathrm{st}}(X_0) = \sum_{i\in I}\left[\sum_{J\subset I}(-1)^{|J|} l\Big((d_i - d_J)\Delta(P)\Big)\right] - \sum_{J\subset I}(-1)^{r-|J|}\left[\sum_{e\in E} l^*(d_J\theta_e)\right] - |P|, \end{gathered} \quad (3.1.6)$$

where $I = \{1,\ldots,r\}$, $d_J := \sum_{j\in J} d_j$ and θ_e is the facet of P corresponding to the edge $e \in E$. The nonzero contributions in the first term of $h^{1,|P|-r-1}_{\mathrm{st}}(X_0)$ comes only from the range of $d_i - d_J \geq 0$ and in the second term from that of $d_J = h_P - 1$ or h_P.

3.2 Constructions via conifold transitions

Let X be a smooth Calabi–Yau 3-fold of Picard number one degenerating to a general complete intersection X_0 in a Gorenstein Hibi toric variety $\mathbb{P}_{\Delta(P)}$ with a finite connected pure poset P. Now we explain the conjectural mirror construction of X via conifold transition proposed by [BCFKvS1] [Bat2].

A *conifold transition* of a smooth Calabi–Yau 3-fold X is the composite operation of a flat degeneration of X to X_0 with finitely many nodes and a small resolution $Y \to X_0$. The conjecture proposed in [BCFKvS1] is that the mirror Calabi–Yau 3-folds Y^* and X^* are again related in the same way. The construction is depicted as the following diagram, Figure 3.2.1. In the diagram, dashed and solid arrows represent flat degenerations and small contraction morphisms, respectively.

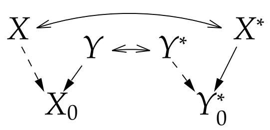


Figure 3.2.1: Mirror symmetry and conifold transitions

In our case, a general complete intersection X_0 in a Gorenstein Hibi toric variety $\mathbb{P}_{\Delta(P)}$ has at worst finitely many nodes because of a Bertini type theorem for toroidal singularities. In fact, we know that three dimensional Gorenstein terminal toric singularities are at worst nodes. Thus we always obtain a conifold transition Y of X which is a smooth Calabi–Yau complete intersection in a MPCP resolution $\widehat{\mathbb{P}}_{\Delta(P)}$ of $\mathbb{P}_{\Delta(P)}$ and can use the Batyrev–Borisov mirror Y^* in § 3.1.

By an argument in [Bat2] on generalized monomial-divisor correspondence, there is a natural specialization Y_0^* of the family of Y^* to get the mirror of X. That is, the specialized parameter $(a_e)_{e\in E}$ should be $\Sigma(\Delta^*)$-admissible, i.e., there exists a $\Sigma(\Delta^*)$-piecewise linear function $\phi : N_{\mathbb{R}} \to \mathbb{R}$ corresponding to a Cartier divisor on X such that $\phi \circ \delta(e) = \log |a_e|$. In all our case, $\operatorname{Pic} \mathbb{P}_{\Delta(P)} \simeq \operatorname{Pic} X \simeq \mathbb{Z}$ holds. Then we can simply specialize the family to be diagonal, i.e., setting all the coefficients a_e to be a same parameter a. Now we repeat the conjecture of [BCFKvS1].

Conjecture 3.2.1 ([BCFKvS1, Conjecture 6.1.2])**.** *Let p be a number of nodes on a Calabi–Yau 3-fold $X_0 \subset \mathbb{P}_{\Delta(P)}$. We define a one parameter family of affine complete intersections in* $(\mathbb{C}^*)^{|P|}$

by the following equations:

$$f_j = 1 - a(\sum_{e\in E_j} t^{\delta(e)}) = 0 \quad \textit{(for all } 1 \le j \le r\textit{).} \tag{3.2.1}$$

The closure Y_0^* of the above set in a MPCP-resolution $\hat{\mathbb{P}}_\nabla$ has p nodes, and there are $(|E|-|P|-p-1)$ relations between the homology classes of p vanishing 3-cycles on Y^* shrinking to nodes in Y_0^* for general a. A small resolution $X^* \to Y_0^*$ is a mirror manifold of X with the correct Hodge numbers, $h^{i,j}(X^*) = h^{3-j,i}(X)$.

Remark 3.2.2. In the case of smoothing of 3-dimensional Calabi–Yau hypersurfaces in Gorenstein Hibi toric varieties, we can see Conjecture 3.2.1 holds by the same argument as in [BCFKvS1] [BK], i.e., in fact the polar duality of faces gives a one-to-one correspondence between singular $\mathbb{P}^1 \subset \mathbb{P}_{\Delta(P)}$ and torus invariant $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}_{\Delta^*}$ which intersect non-transversally with the closure of the set $\{f_1 = 0\}$. Further the MPCP-resolution $\hat{\mathbb{P}}_{\Delta^*} \to \mathbb{P}_{\Delta^*}$ increases them by $h_P = \deg X$ times.

In general, the existence of a smooth mirror X^* is still an open problem. In the remaining part, we refer to not only X^* but also Y_0^* as a *conjectural mirror* of X.

3.3 Fundamental period

We derive the explicit form of the fundamental period for the conjectural mirror family of X. Obviously, the coordinate transformation $t_u \to \zeta^{h_u} t_u$ gives a $\mathbb{Z}_{h_P}$-symmetry $a \to \zeta a$ in the family in Conjecture 3.2.1, where $\zeta = e^{2\pi\sqrt{-1}/h_P}$. Therefore we should take $x := a^{h_P}$ as a genuine moduli parameter. The fundamental period $\omega_0(x)$ of the mirror family is defined by integration of the holomorphic $(|P|-r)$-form Ω_x on a (real) torus cycle $\mathbb{T}$ that vanishes at $x = 0$. By residue theorem, we get the following formula up to the constant multiplication,

$$\begin{aligned}
\omega_0(x) &= \int_{\mathbb{T}} \Omega_x = \frac{1}{(2\pi\sqrt{-1})^{|P|}} \int_{|t_u|=1} \frac{1}{\prod_{j=1}^r f_j} \prod_{i=1}^{|P|} \frac{\mathrm{d}t_i}{t_i} \\
&= \sum_{m=0}^{\infty} a^{h_P m} \frac{1}{(2\pi\sqrt{-1})^{|P|}} \int_{|t_u|=1} \prod_{j=1}^{r} (\sum_{e\in E_j} t^{\delta(e)})^{d_j m} \prod_{i=1}^{|P|} \frac{\mathrm{d}t_i}{t_i} \\
&= \sum_{m=0}^{\infty} x^m \# \left\{ \phi : \bigcup_{j=1}^{r} J_j(m) \to E \;\middle|\; \phi(J_j(m)) \subset E_j, \sum_{s(e)=u} \phi^{-1}(e) = \sum_{t(e)=u} \phi^{-1}(e) \right\} \\
&= \sum_{m=0}^{\infty} x^m \frac{\prod_{j=1}^r (d_j m)!}{m!^{h_P}} \# \left\{ \psi : \bigcup_{k=1}^{h_P} J^k(m) \to E \;\middle|\; \psi(J^k(m)) \subset E^k, \sum_{s(e)=u} \psi^{-1}(e) = \sum_{t(e)=u} \psi^{-1}(e) \right\},
\end{aligned}$$

where $J_j(m) := \left\{(j,i) \in \mathbb{N}^2 \mid 1 \le i \le d_j m\right\}$ and $J^k(m) := \{(k,i) \in \mathbb{N}^2 \mid 1 \le i \le m\}$.

In the case that the Hasse diagram of P (and hence $\hat{P}$) is a plane graph, we can go further like [BCFKvS2]. This is originally formulated in the work of Bondal and Galkin [BG] for the Landau–Ginzburg mirror of minuscule homogeneous space G/Q (cf. § 4). If the Hasse diagram of P is a plane graph, we can define the dual graph B of the Hasse diagram of $\hat{P}$ on a sphere $S^2 = \mathbb{P}^1$ with putting $\hat{1}, \hat{0}$ on $\pm\sqrt{-1}\infty$ respectively. We denote by b_L, b_R the elements $b \in B$ corresponding to the farthest left and right areas respectively. We draw the Hasse diagram of $\hat{P}$ and its dual graph B below, Figure 3.3.1, for the minuscule poset of $G(2,6)$ as an example (cf. § 4).

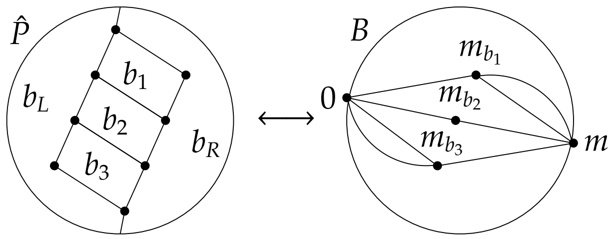


Figure 3.3.1: An example of $\hat{P}$ and the dual graph B

The orientation of an edge e of B is defined as the direction from the left $l(e)$ to the right $r(e)$. We attain the variable m_b for each element $b \in B$ and set $m_{b_L} = 0$ and $m_{b_R} = m$.

Proposition 3.3.1. *Let X be a smooth Calabi–Yau variety of Picard number one degenerating to a general complete intersection X_0 in a Gorenstein Hibi toric variety $\mathbb{P}_{\Delta(P)}$ with a finite connected pure poset P. Assume that the Hasse diagram of P is a plane graph. Then, the fundamental period $\omega_0(x)$ for the conjectural mirror family of X is presented in the following:*

$$\omega_0(x) = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{r}(d_j m)!}{m!^{h_P}} \sum_{m_b \in B} \prod_{e \in E(B)} \binom{m_{r(e)}}{m_{l(e)}} x^m. \tag{3.3.1}$$

3.4 Assumptions from mirror symmetry

We prepare some further definitions related with the monodromy calculations of a Picard–Fuchs operator $\mathcal{D}_x$ in one variable x which has two maximally unipotent monodromy (MUM) point at $x = 0$ and $x = \infty$. Assume that there exist smooth Calabi–Yau 3-folds X and Z in the mirror side associated with the MUM points $x = 0, \infty$, respectively.

Recall the argument in [CdOGP] based on the mirror symmetry, which involves an integral symplectic basis of solutions in the original Calabi–Yau geometry. Let us start from the Frobenius basis of solutions for $\mathcal{D}_x$ around $x = 0$, namely the unique normalized regular power series solutions $\omega_0(x) = 1 + O(x)$ and the followings

$$\begin{aligned}
\omega_1(x) &= \omega_0(x)\log x + \omega_1^{\mathrm{reg}}(x),\\
\omega_2(x) &= \omega_0(x)(\log x)^2 + 2\omega_1^{\mathrm{reg}}(x)\log x + \omega_2^{\mathrm{reg}}(x),\\
\omega_3(x) &= \omega_0(x)(\log x)^3 + 3\omega_1^{\mathrm{reg}}(x)(\log x)^2 + 3\omega_2^{\mathrm{reg}}(x)\log x + \omega_3^{\mathrm{reg}}(x),
\end{aligned} \tag{3.4.1}$$

where $\omega_k^{\mathrm{reg}}(x)$ is a regular power series around $x = 0$ without constant term. We expect an integral symplectic basis has the following form:

$$\Pi^X(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \beta/24 & a & -\kappa/2 & 0 \\ \gamma & \beta/24 & 0 & \kappa/6 \end{pmatrix} \begin{pmatrix} n_0\omega_0(x) \\ n_1\omega_1(x) \\ n_2\omega_2(x) \\ n_3\omega_3(x) \end{pmatrix}, \tag{3.4.2}$$

where $\kappa = -\deg(X)$, $\beta = -c_2(X)\cdot H$, $\gamma = -n_3\zeta(3)\chi(X)$, $n_k = 1/(2\pi i)^k$ with the topological invariants of X (cf. §4.6), and a is an unknown parameter without geometric interpretation although it may be consistent to choose $a \in \deg(X)/2 + \mathbb{Z}$.

Around $x = \infty$, we also expect the existence of similar basis $z^\rho \Pi^Z(z)$ of solutions for $\mathcal{D}_z$ under some appropriate coordinate change $z = c/x$, where ρ is the index of the singularity at $x = \infty$ of $\mathcal{D}_x$. We denote by $\Pi^Z(z)$ the gauge fixed basis, exactly the same form as $\Pi^X(x)$ with the Frobenius basis $\omega_k^Z(z)$ for $\mathcal{D}_z^Z := z^\rho \mathcal{D}_{c/z} z^{-\rho}$ and topological invariants $\deg(Z)$, $c_2(Z)\cdot H$, $\chi(Z)$ and an unknown parameter a^Z.

4 Complete Intersections in Minuscule Schubert Varieties

In this chapter, we study complete intersection Calabi–Yau 3-folds in minuscule Schubert varieties including a new example $\mathbf{\Sigma}(1^9)$.

From Section 4.1 to Section 4.4 are devoted to give preliminaries for the combinatorial notion of minuscule posets and the geometry of minuscule Schubert varieties.

In Section 4.5, we make a list of all the deformation equivalent (diffeomorphic) classes of smooth complete intersection Calabi–Yau 3-folds in minuscule Schubert varieties. We will see that there is a unique nontrivial example of such Calabi–Yau 3-folds, $\mathbf{\Sigma}(1^9)$ embedded in a locally factorial Schubert variety $\mathbf{\Sigma}$ in the Cayley plane $\mathbb{OP}^2$.

In Section 4.6, we give a computational method of calculating topological invariants for a smooth Calabi–Yau 3-folds of Picard number one degenerating to a general complete intersection in a Gorenstein Hibi toric variety. We work on $\mathbf{\Sigma}(1^9)$ as an example. The topological Euler number is computed by using a conifold transition.

In Section 4.7, we study the mirror symmetry of $\mathbf{\Sigma}(1^9)$ using the results and the assumptions in Chapter 3. We obtain the Picard–Fuchs operator $\mathcal{D}_x$, which suggests the existence of a non-trivial Fourier–Mukai partner of X (Conjecture 4.7.2). We also perform the monodromy calculation. Every result seems very similar to that happened for the examples of the Pfaffian-Grassmannian [Rød] [HK] and the Reye congluence Calabi–Yau 3-fold [HT1, HT2].

4.1 Definitions

First of all, let us define a minuscule weight, a minuscule homogeneous space and minuscule Schubert varieties (Definition 4.1.1). Let G be a simply connected simple complex algebraic group, B a Borel subgroup and T a maximal torus in B. We denote by R^+ the set of positive roots and by $S = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots. Let W be the Weyl group of G. Denote by Λ the character group of T, also called the weight lattice of G. The weight lattice Λ is generated by the fundamental weights $\lambda_1, \ldots, \lambda_n$ defined by $(\alpha_i^\vee, \lambda_j) = \delta_{ij}$ for $1 \le i, j \le n$, where $(,)$ is a W-invariant inner product and $\alpha^\vee := 2\alpha/(\alpha, \alpha)$. An integral weight $\lambda = \sum n_i \lambda_i \in \Lambda$ is said to be dominant if $n_i \ge 0$ for all $i = 1, \ldots, n$. For an integral dominant weight $\lambda \in \Lambda$, we denote by V_λ the irreducible G-module of the highest weight λ. The associated homogeneous space G/Q of λ is the G-orbit of the highest weight vector in the projective space $\mathbb{P}(V_\lambda)$, where $Q \supset B$ is the associated parabolic subgroup of G. A Schubert variety in G/Q is the closure of a B-orbit in G/Q.

Definition 4.1.1 (cf. [LMS, Definition 2.1])**.** Let $\lambda \in \Lambda$ be a fundamental weight. We call λ *minuscule* if it satisfies the following equivalent conditions.

(1) Every weight of V_λ is in the orbit $W\lambda \subset \Lambda$.

(2) $(\alpha^\vee, \lambda) \le 1$ for all $\alpha \in R^+$.

The homogeneous space G/Q associated with a minuscule weight λ is said to be minuscule. The Schubert varieties in minuscule G/Q are also called minuscule.

4.2 Minuscule homogeneous spaces

We give some further notations and recall the classification of minuscule homogeneous spaces (Table 4.2.1). A parabolic subgroup $Q \supset B$ is determined by a subset S_Q of S associated with negative root subgroups. A useful notation for a homogeneous space G/Q is to cross the nodes in the Dynkin diagram which correspond to the simple roots in $S \setminus S_Q$. With this notation, the minuscule homogeneous spaces are as shown in Table 4.2.1. This contains the Grassmannians $G(k, n)$, the orthogonal Grassmannians $OG(n, 2n)$, even dimensional quadrics Q^{2n} and, finally, the Cayley plane $\mathbb{OP}^2 = E_6/Q_1$ and the Freudenthal variety E_7/Q_7, where we use the Bourbaki labelling for the roots. We omit two kinds of minuscule weights for groups of type B and type C, since they give the isomorphic varieties to those for simply laced groups.

$G(k, n+1)$	$OG(n, 2n)$	Q^{2n-2}	$\mathbb{OP}^2$	E_7/Q_7

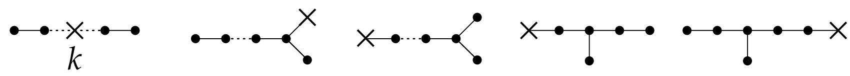


Table 4.2.1: Minuscule homogeneous spaces

4.3 Minuscule posets

The Weyl group W is generated by simple reflections $s_\alpha \in W$ for $\alpha \in S$. These generators define the length function l on W. Let us denote by W_Q the Weyl group of Q, i.e. the subgroup generated by $\{s_\alpha \in W \mid \alpha \in S_Q\}$, and by W^Q the set of minimal length representatives of the coset W/W_Q in W. For any $w \in W^Q$, we denote by $X(w) = \overline{BwQ/Q}$ the Schubert variety in G/Q associated with w, which is a $l(w)$-dimensional normal Cohen–Macaulay projective variety with at worst rational singularities. There is a natural partial order $\prec$ on W^Q called the Bruhat order, defined as $w_1 \leq w_2 \Leftrightarrow X(w_1) \subset X(w_2)$. We recall the following fundamental fact for minuscule homogeneous spaces.

Proposition 4.3.1 ([Pro, Proposition V.2]). *For a minuscule homogeneous space G/Q, the poset W^Q is a finite distributive lattice.*

From Proposition 4.3.1 and the Birkhoff representation theorem, Theorem 1.2.2, we can define the *minuscule poset* P_Q for a minuscule G/Q such that $J(P_Q) = W^Q$ as in [Pro]. Moreover, the order ideal $P_w \subset P_Q$ associated with $w \in W^Q$ is called the minuscule poset for the minuscule Schubert variety $X(w) \subset G/Q$. For example, the order ideals $P_Q, \varnothing \subset P_Q$ turn out to be the minuscule posets for the total space $X(w_Q) = G/Q$ and the B-fixed point $X(\mathrm{id}) = Q/Q$, respectively, where w_Q is the unique longest element in W^Q. The minuscule poset for minuscule Schubert varieties is a generalization of the Young diagram for Grassmann Schubert varieties.

Example 4.3.2. An easy method to compute the Hasse diagram of W^Q to trace out the W-orbit of certain dominant weight whose stabilizer coincides with W_Q (cf. [BE, §4.3]). Denote by $(ij \cdots k)$ the element $w = s_{\alpha_i} s_{\alpha_j} \cdots s_{\alpha_k} \in W$, where s_α is simple reflection with respect to $\alpha \in S$. The initial part of the Hasse diagram of W^Q for the Cayley plane $\mathbb{OP}^2$ is the following

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad\qquad\qquad\quad (65431) \\
\qquad\qquad\qquad\qquad\quad (5431) \diagup \qquad\quad \diagdown (265431) \\
\mathrm{id} - (1) - (31) - (431) \langle \qquad\qquad \rangle (25431) \langle \qquad\qquad \rangle (4265431) \cdots \\
\qquad\qquad\qquad\qquad\quad (2431) \diagup \qquad\quad\qquad \diagdown (425431) \diagdown \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3425431) \cdots ,
\end{array}
$$

where the right covers the left for connected two elements with respect to the Bruhat order. Thus we obtain the Hasse diagram of the distributive lattice W^Q and hence the minuscule poset for every Schubert variety in $\mathbb{OP}^2$.

Definition 4.3.3. In the above notations, let us set $w = (345134265431) \in W^Q$. We denote by $\boldsymbol{\Sigma}$ the associated 12-dimensional Schubert variety $X(w)$ in the Cayley plane $\mathbb{OP}^2$ corresponding to the minuscule poset P in Figure 1.2.1.

Remark 4.3.4. We remark on another geometric characterization of minuscule homogeneous spaces in [LMS, Definition 2.1]. A fundamental weight λ is minuscule (Definition 4.1.1) if and only if the following condition holds.

(3) For the associated homogeneous space G/Q, the Chevalley formula

$$[H] \cdot [X(w)] = \sum_{w \text{ covers } w'} [X(w')] \tag{4.3.1}$$

holds for all $w \in W^Q$ in the Chow ring of G/Q, where H is the unique Schubert divisor in G/Q and W^Q is the poset with the Bruhat order.

As a corollary, it turns out that the degree of a minuscule Schubert variety $X(w)$ with respect to $\mathcal{O}_{G/Q}(1)|_{X(w)}$ equals the number of maximal chains in $J(P_w)$. For example, we obtain $\deg \boldsymbol{\Sigma} = 33$ by counting the maximal chains in $J(P)$ in Figure 1.2.1.

4.4 Singularities

We introduce further definitions to describe singularities of minuscule Schubert varieties. As we expect from the computation in Example 4.3.2, the Bruhat order on W^Q is generated by simple reflections for minuscule G/Q [LW, Lemma 1.14], that is,

$$w_1 \text{ covers } w_2 \Leftrightarrow w_1 = s_\alpha \cdot w_2 \text{ and } l(w_1) = l(w_2) + 1 \text{ for some } \alpha \in S.$$

From this fact, a join irreducible element $u \in W^Q$ covers the unique element $s_{\beta_Q(u)} \cdot u \in W^Q$ where $\beta_Q(u) \in S$. Thus we can define the natural coloration $\beta_Q : P_Q \to S$ for a minuscule poset P_Q by simple roots S. We also define the coloration β_w on each minuscule poset $P_w \subset P_Q$ by restricting β_Q on P_w. The minuscule poset P_w with the coloration

$\beta_w : P_w \to S$ has in fact the same information as the minuscule quiver introduced by Perrin [Per1, Per2], which gives a good description of geometric properties of minuscule Schubert varieties $X(w)$. Now we translate the combinatorial notions and useful facts on singularities of minuscule Schubert varieties $X(w)$ from [Per1, Per2] in our terminology.

Definition 4.4.1. Let P be a minuscule poset with the coloration $\beta : P \to S$.

(1) A *peak* of P is a maximal element u in P.

(2) A *hole* of P is a maximal element u in $\beta^{-1}(\alpha)$ for some $\alpha \in S$ such that there are exactly two elements $v_1, v_2 \in P$ with $u < v_i$ and $(\beta(u)^\vee, \beta(v_i)) \neq 0$ $(i = 1, 2)$.

Let us denote by Peaks(P) and Holes(P) the set of peaks and holes of P, respectively. A hole u of the poset P is said to be *essential* if the order ideal $P^u := \{v \in P \mid v \not\geq u\}$ contains all other holes in P.

Let $X(w)$ be a minuscule Schubert variety in G/Q and P_w the associated minuscule poset. Weil and Cartier divisors on $X(w)$ are described in terms of the poset P_w. In fact, it is clear that any Schubert divisor coincides with a Schubert variety D_u associated with P_w^u for some $u \in \mathrm{Peaks}(P_w)$. It is well-known that the divisor class group $\mathrm{Cl}(X(w))$ is the free $\mathbb{Z}$-module generated by the classes of the Schubert divisors D_u for $u \in \mathrm{Peaks}(P_w)$, and the Picard group $\mathrm{Pic}(X(w))$ is isomorphic to $\mathbb{Z}$ generated by $\mathcal{O}_{G/Q}(1)|_{X(w)}$. As we saw in Remark 4.3.4, the Cartier divisor corresponding to $\mathcal{O}_{G/Q}(1)|_{X(w)}$ is

$$\sum_{u \in \mathrm{Peaks}(P_w)} D_u. \tag{4.4.1}$$

Proposition 4.4.2 ([Per1, Per2])**.** *Let $X(w)$ be a minuscule Schubert variety and P_w the associated minuscule poset.*

(1) [Per1, Proposition 4.17] An anticanonical Weil divisor of $X(w)$ is

$$-K_{X(w)} = \sum_{u \in \mathrm{Peaks}(P_w)} (h(u) + 1) D_u. \tag{4.4.2}$$

In particular, $X(w)$ is Gorenstein if and only if P_w is pure. In this case $X(w)$ is a Fano variety of index h_{P_w}.

(2) [Per2, Theorem 2.7 (1)] The Schubert subvariety associated with the order ideal $P_w^u \subset P_w$ for an essential hole u of P_w is an irreducible component of the singular loci of $X(w)$. All the irreducible components of the singular loci are obtained in this way.

We apply Proposition 4.4.2 to our example $\Sigma \subset \mathbb{OP}^2$ and obtain the following.

Proposition 4.4.3. *Let* Σ *be the minuscule Schubert variety in* $\mathbb{OP}^2$ *(Definition 4.3.3).*

(1) Σ *is a locally factorial Gorenstein Fano variety of index 9.*

(2) The singular locus of Σ *is isomorphic to* $\mathbb{P}^5$*.*

Proof. The former holds because the corresponding minuscule poset P (Figure 1.2.1) is pure with $h_P = 9$ and the unique peak. From the computation of the Hasse diagram of W^Q of $\mathbb{OP}^2$ in Example 4.3.2, the coloration $\beta : P \to S$ is given as the following picture.

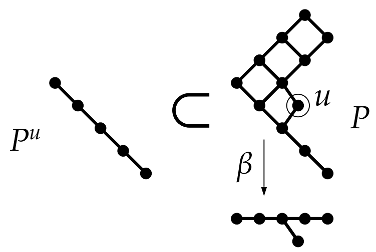


Figure 4.4.1: The coloration of the minuscule poset P and the singular locus

A unique (essential) hole of P is the circled vertex u, whose color is $\alpha_2 \in S$. The corresponding Schubert subvariety is described by the minuscule poset P^u, which coincides with the singular locus of Σ by Proposition 4.4.2. It is isomorphic to $\mathbb{P}^5$ because the degree equals to one. □

We record the useful vanishing theorems for minuscule Schubert varieties.

Theorem 4.4.4 ([LMS, Theorem 7.1])**.** *Let* λ *be a minuscule weight,* $G/Q \subset \mathbb{P}(V_\lambda)$ *the associated homogeneous space and* $X(w) \subset G/Q$ *a minuscule Schubert variety.*

(1) $H^0(\mathbb{P}(V_\lambda), \mathcal{O}(m)) \to H^0(X(w), \mathcal{O}(m))$ *is surjective for all* $m \geq 0$,

(2) $H^i(X(w), \mathcal{O}(m)) = 0$ *for all* $m \in \mathbb{Z}$ *and* $0 < i < l(w)$,

(3) $H^{l(w)}(X(w), \mathcal{O}(m)) = 0$ *for all* $m \geq 0$.

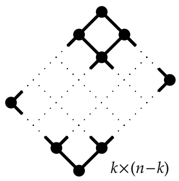
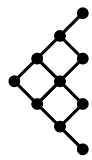
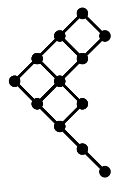
4.5 List of complete intersection Calabi–Yau 3-folds

Now we study the smooth complete intersection Calabi–Yau 3-folds in minuscule Schubert varieties. We show that there is a unique new deformation equivalent class of such Calabi–Yau 3-folds, that is, the complete intersection of nine hyperplanes in a locally factorial Schubert variety $\mathbf{\Sigma}$ in Definition 4.3.3.

First we fix some basic terminologies to clarify the meaning of our list. Let $X(w)$ be a minuscule Schubert variety. We call a subvariety $X \subset X(w)$ a *complete intersection* if it is the common zero locus of $r = \mathrm{codim} X$ global sections of invertible sheaves on $X(w)$. We may denote by $X = X(w)(d_1, \ldots, d_r)$ the complete intersection variety of general r sections of degree $d_1, \ldots, d_r$ with respect to $\mathcal{O}_{G/Q}(1)|_{X(w)}$ since $\mathrm{Pic}\, X(w) \simeq \mathbb{Z}$. A *Calabi–Yau* variety X is a normal projective variety with at worst Gorenstein canonical singularities and with trivial canonical bundle $K_X \simeq 0$ such that $H^i(X, \mathcal{O}_X) = 0$ for all $0 < i < \dim X$. Two smooth varieties X_1 and X_2 are called *deformation equivalent* if there exist a smooth family $\mathcal{X} \to U$ over a connected open base $U \subset \mathbb{C}$ such that $\mathcal{X}_{t_1} \simeq X_1$ and $\mathcal{X}_{t_2} \simeq X_2$ for some $t_1, t_2 \in U$. In this case, X_1 and X_2 turn out to be diffeomorphic.

We summarize all possible smooth complete intersection Calabi–Yau 3-folds in minuscule Schubert varieties:

Proposition 4.5.1. *A smooth complete intersection Calabi–Yau 3-fold in a minuscule Schubert variety is one of that listed in the following table up to deformation equivalences.*

minuscule posets			
ambient varieties	$G(k,n)$	$OG(5,10)$	$\mathbf{\Sigma}$
degrees	*10 examples*	$(1^6, 2)$	(1^9)

In this table, 10 known examples in Grassmannians of type A include five in projective spaces;

$$\mathbb{P}^4(5),\ \mathbb{P}^5(2,4),\ \mathbb{P}^5(3^2),\ \mathbb{P}^6(2^2,3) \text{ and } \mathbb{P}^7(2^4),$$

and five in others, whose mirror symmetry was discussed in [BCFKvS1];

$$G(2,5)(1^2,3),\ G(2,5)(1,2^2),\ G(2,6)(1^4,2),\ G(3,6)(1^6) \text{ and } G(2,7)(1^7).$$

For all these Calabi–Yau 3-folds, the Picard number equals to one.

Proof. We may assume that the ambient minuscule Schubert variety is Gorenstein. In fact, from the adjunction formula and the Grothendieck–Lefschetz theorem for divisor class groups of normal projective varieties [RS], we have an explicit formula of the canonical divisor as a Cartier divisor, $K_{X(w)} = -D_1 - \cdots - D_r$ where $D_j \subset X(w)$ is a very ample Cartier divisor of degree d_j and $X(w)(d_1, \ldots, d_r)$ is a general Calabi–Yau complete intersection.

Let $P^* = P \cup \{\hat{1}\}$, $P_* = P \cup \{\hat{0}\}$ be the posets with the partial orders extended from a finite poset P by $u \prec \hat{1}$, $\hat{0} \prec u$ for all $u \in P$, respectively (cf. Example 1.2.3). From Corollary 1.3.5, d-times iterated extension $P^{*\cdots*}_{*\cdots*}$ corresponds to d-times iterated projective cones over $\mathbb{P}_{\Delta(P)}$. Let $X(w)$, $X(w')$ be minuscule Schubert varieties and P_w, $P_{w'}$ the corresponding minuscule posets, respectively. Assume that $P_{w'}$ coincides with a d-times iterated extension $(P_w)^{*\cdots*}_{*\cdots*}$ of P_w. It holds that $\mathbb{P}_{\Delta(P_w)}$ is isomorphic to a complete intersection of d general hyperplanes in $\mathbb{P}_{\Delta(P_{w'})}$. By Theorem 2.3.2, there exist toric degenerations of $X(w)$ and $X(w')$ to the Hibi toric varieties $\mathbb{P}_{\Delta(P_w)}$ and $\mathbb{P}_{\Delta(P_{w'})}$, respectively. This means that general complete intersection Calabi–Yau 3-folds $X = X(w)(d_1, \ldots, d_r)$ and $X' = X(w')(1^d, d_1, \ldots, d_r)$ can be connected by flat deformations through a complete intersection $X_0 = \mathbb{P}_{\Delta(P_w)}(d_1, \ldots, d_r)$. Since X_0 has at worst terminal singularities, the Kuranishi space is smooth by [Nam, Theorem A] and the degenerating loci have a positive complex codimension. Therefore X and X' are connected by smooth deformation. Thus we eliminate redundancy arisen from iterated extensions of minuscule posets.

A Gorenstein minuscule Schubert variety $X(w)$ with minuscule poset $P = P_w$ is a $|P|$-dimensional Fano variety of index h_P as we saw in Proposition 4.4.2 (1). The condition for general complete intersections in $X(w)$ to be Calabi–Yau 3-folds gives a strong combinatorial restriction for the poset P as follows,

$$h_P - 1 \leq |P| \leq h_P + 3. \tag{4.5.1}$$

On the other hand, there is a complete list of the minuscule posets in [Per1]. Hence we can make a list of the complete intersection Calabi–Yau 3-folds by counting such posets.

We check that the resulting 3-folds $X \subset X(w)$ with trivial canonical bundles turn out to be Calabi–Yau varieties after some computation using the vanishing theorems for $X(w)$, Theorem 4.4.4. We verify the smoothness of these 3-folds by looking at the codimension of the singular loci of $X(w)$ using Proposition 4.4.2 (2). For example, a general linear section $X = \mathbf{\Sigma}(1^9)$ is smooth since the singular loci of $\mathbf{\Sigma}$ have codimension 7 as we saw in Proposition 4.4.3. All the smooth cases are contained in locally factorial minuscule Schubert varieties, i.e., the minuscule poset P has the unique peak. Thus the Picard number equals to one again by the Grothendieck–Lefschetz theorem for divisor class groups [RS]. This completes the proof. □

4.6 Topological invariants

Now we explain our calculation of topological invariants valid for a smooth Calabi–Yau 3-fold of Picard number one degenerating to a general complete intersection Calabi–Yau 3-fold in a Gorenstein Hibi toric variety by taking $X = \mathbf{\Sigma}(1^9)$ as an example. The topological invariants mean the degree $\deg(X) = \int_X H^3$, the linear form associated with the second Chern class $c_2(X) \cdot H = \int_X c_2(X) \cup H$ and the topological Euler number $\chi(X) = \int_X c_3(X)$, where H is the ample generator of $\operatorname{Pic}(X) \simeq \mathbb{Z}$. These three invariants characterize the diffeomorphic class of smooth simply connected Calabi–Yau 3-folds of Picard number one [Wal].

Proposition 4.6.1. *The topological invariants of $X = \mathbf{\Sigma}(1^9)$ are*

$$\deg(X) = 33, \quad c_2(X) \cdot H = 78, \quad \chi(X) = -102.$$

Proof. The degree of X coincides with that of the minuscule Schubert variety $\mathbf{\Sigma} \subset \mathbb{OP}^2$ since the ample generator $\mathcal{O}_{\mathbf{\Sigma}}(1)$ of $\operatorname{Pic} \mathbf{\Sigma}$ is the restriction of $\mathcal{O}_{\mathbb{OP}^2}(1)$ and X is a linear section. We obtain $\deg(\mathbf{\Sigma}) = 33$ by using the Chevalley formula of $\mathbb{OP}^2$ as we already saw in Remark 4.3.4.

The Schubert variety $V^0 := \mathbf{\Sigma}$ and its general complete intersections $V^j := \mathbf{\Sigma}(1^j)$ have at worst rational singularities. Hence the Kawamata–Viehweg vanishing theorem gives

$$H^i(V^j, \omega_{V^j} \otimes \mathcal{O}_{V^j}(k)) = H^i(V^j, \mathcal{O}_{V^j}(k + j - 9)) = 0 \quad \text{for all } i > 0 \text{ and } k > 0. \tag{4.6.1}$$

Together with the long cohomology exact sequences of

$$0 \to \mathcal{O}_{V^j}(k) \to \mathcal{O}_{V^j}(k+1) \to \mathcal{O}_{V^{j+1}}(k+1) \to 0,$$

the holomorphic Euler number of $X = V^9$ becomes

$$\chi(X, \mathcal{O}_X(1)) = \dim H^0(X, \mathcal{O}_X(1)) = \dim H^0(\mathbf{\Sigma}, \mathcal{O}_{\mathbf{\Sigma}}(1)) - 9 = |J(P)| - 9 = 12. \tag{4.6.2}$$

On the other hand, it holds that

$$\chi(X, \mathcal{O}_X(1)) = \frac{1}{6}\deg(X) + \frac{1}{12} c_2(X) \cdot H$$

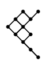
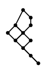
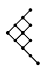
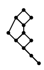
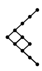
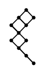
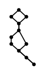
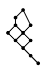
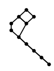
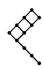
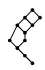
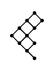
from the Hirzebruch–Riemann–Roch theorem of the smooth Calabi–Yau 3-fold X. Thus we obtain $c_2(X) \cdot H = 78$.

For the topological Euler number $\chi(X)$, we use the toric degeneration of $\mathbf{\Sigma}$ to the Hibi toric variety $\mathbb{P}_{\Delta(P)}$, Theorem 2.3.2. Recall that we have a conifold transition Y

of X passing through a general complete intersection Calabi–Yau 3-fold X_0 in the degenerated variety $\mathbb{P}_{\Delta(P)}$ and a MPCP resolution $\widehat{\mathbb{P}}_{\Delta(P)}$ of $\mathbb{P}_{\Delta(P)}$ (cf. § 3.2). By Theorem 3.1.1, the Hodge numbers of Y can be calculated as $h^{1,1}(Y) = 5$ and

$$\begin{aligned} h^{2,1}(Y) &= 9\,(|J(P)| - 9) - \sum_{e\in E}(l^*(9\theta_e) - 9l^*(8\theta_e)) - |P| \\ &= 96 - \sum_{e\in E}(l^*(9\theta_e) - 9l^*(8\theta_e)). \end{aligned} \tag{4.6.3}$$

To count the number of interior integral points in each facet, we use Proposition 1.4.3 which states a face of the order polytope is also the order polytope of some poset P'. For each facet θ_e, the corresponding poset P' (or $\hat{P}'$) is easily obtained by replacing an inequality $x_{s(e)} \geq x_{t(e)}$ by the equality $x_{s(e)} = x_{t(e)}$, and by considering the induced partial order. The Hasse diagram of resulting posets P' are shown in the following table, where the numbering of edges is chosen from the upper left in a picture of the Hasse diagram of $\hat{P}$ in Figure 1.2.1.

facets	θ_1	θ_2	θ_3, θ_6	θ_4	θ_5	θ_7, θ_{10}	θ_8	θ_9	θ_{11}	θ_{12}, θ_{14}	θ_{13}	$\theta_{15}, \theta_{16}, \theta_{17}$
												
$l^*(8\theta_i)$	1	-	-	-	-	-	-	-	-	-	-	1
$l^*(9\theta_i)$	20	3	1	2	-	1	-	2	-	1	2	20

As shown in this table, some posets P' are pure and others are not pure. For a pure poset P', the face $\theta_e \simeq \Delta(P')$ is unimodular equivalent to a reflexive polytope (cf. § 1.6). Then we know $l^*(h_{P'}\theta_e) = 1$ and $l^*((h_{P'} + 1)\theta_e) = |J(P')|$. When P' is not pure, we can also easily obtain the number $l^*(k\theta_e)$ by counting the points satisfying the inequalities of the polytope $k\theta_e \simeq k\Delta(P')$ strictly. For example, $9\theta_2$ contains three internal integral points corresponding to

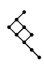
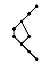
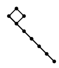
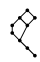
$$\begin{array}{c} 9 \\ x_1\;x_2 \\ x_3\;x_4 \\ x_5\;x_6 \\ x_7\;x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ 0 \end{array} = \begin{array}{c} 9 \\ 8\;7 \\ 6\;6 \\ 5\;5 \\ 4\;4 \\ 3 \\ 2 \\ 1 \\ 0 \end{array}, \quad \begin{array}{c} 9 \\ 8\;7 \\ 7\;6 \\ 5\;5 \\ 4\;4 \\ 3 \\ 2 \\ 1 \\ 0 \end{array} \quad \text{and} \quad \begin{array}{c} 9 \\ 8\;7 \\ 7\;6 \\ 6\;5 \\ 4\;4 \\ 3 \\ 2 \\ 1 \\ 0 \end{array}$$

From the table, we get $h^{2,1}(Y) = 37$, hence $\chi(Y) = 2(h^{1,1}(Y) - h^{2,1}(Y)) = -64$.

Let us recall that the conifold transition is a surgery of Calabi–Yau 3-folds replacing finite vanishing S^3 by the same number of exceptional $\mathbb{P}^1 \simeq S^2$ as in [Cle]. From the inclusion-exclusion principle of the Euler numbers, $\chi(X)$ and $\chi(Y)$ are related with each other as:

$$\chi(X) = \chi(Y) - 2p, \tag{4.6.4}$$

where p is the number of nodes on X_0. Then we need to know the total degree of codimension three singular loci of $\mathbb{P}_{\Delta(P)}$, which equals the number $p/\prod_j d_j = p$ in our case. From Proposition 1.5.1 (2), an irreducible component of singular loci of the Hibi toric variety $\mathbb{P}_{\Delta(P)}$ corresponds to a minimal convex cycle in $\hat{P}$. There are four such cycles (or boxes) $b_1, \cdots, b_4$ and all of them define the codimension three faces in $\Delta(P)$ as in Proposition 1.4.3. Again we can compute the corresponding index poset P' of them by the method used above. The resulting posets P' are summarized as follows.

singular loci	b_1	b_2	b_3	b_4
				
degree	5	3	2	9

From Proposition 1.5.1 (3), we can compute the degree of each irreducible component of singular loci by counting the maximal chains in $J(P')$. Then we obtain that total degree p, that is, the number of nodes on X_0 is 19. We conclude $\chi(X) = -102$. □

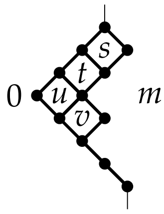
Remark 4.6.2. 1. The existence of the Calabi–Yau 3-fold with these topological invariants were previously conjectured by [vEvS] from the monodromy calculations of Calabi–Yau differential equations. We also perform the similar calculation in the next section.

2. It may be possible to calculate the topological Euler number $\chi(X)$ in another way, by computing the Chern–Mather class of the Schubert variety $\mathbf{\Sigma}$. For the Grassmann Schubert varieties, this is done by [Jon] using Zelevinsky's *IH*-small resolution. In our case, however, it is known that $\mathbf{\Sigma}$ does not admit any *IH*-small resolution [Per1].

4.7 Mirror symmetry for $\mathbf{\Sigma}(1^9)$

From Theorem 2.3.2, we have a toric degeneration of $\mathbf{\Sigma}$ to the Hibi toric variety $\mathbb{P}_{\Delta(P)}$ where P is the minuscule poset for $\mathbf{\Sigma}$. Thus we can use all the results in § 3 based on the conjectural mirror construction via conifold transition.

The fundamental period of the conjectural mirror family of X can be read from the following diagram.



The vertices of the dual graph B corresponds to the separated areas. The fundamental period turns out to be

$$\omega_0(x) = \sum_{m,s,t,u,v} \binom{m}{s}^2 \binom{m}{v}^2 \binom{m}{t}\binom{s}{t}\binom{t}{u}\binom{v}{u} x^m, \tag{4.7.1}$$

where $x = a^9$. With the aid of numerical method, we obtain the Picard–Fuchs equation for the conjectural mirror family of X.

Proposition 4.7.1. *Let $\omega_0(x)$ be the above power series around $x = 0$, which corresponds to the fundamental period for the conjectural mirror family of the Calabi–Yau 3-fold $X = \mathbf{\Sigma}(1^9)$. This satisfies the Picard–Fuchs equation $\mathcal{D}_x\omega_0(x) = 0$ with $\theta_x = x\partial_x$ and*

$$\begin{aligned}\mathcal{D}_x =& 121\theta_x^4 - 77x(130\theta_x^4 + 266\theta_x^3 + 210\theta_x^2 + 77\theta_x + 11)\\ &- x^2(32126\theta_x^4 + 89990\theta_x^3 + 103725\theta_x^2 + 55253\theta_x + 11198)\\ &- x^3(28723\theta_x^4 + 74184\theta_x^3 + 63474\theta_x^2 + 20625\theta_x + 1716)\\ &- 7x^4(1135\theta_x^4 + 2336\theta_x^3 + 1881\theta_x^2 + 713\theta_x + 110) - 49x^5(\theta_x + 1)^4.\end{aligned}$$

The Riemann scheme of the differential operator $\mathcal{D}_x$ is

$$P\left\{\begin{array}{cccccc} \zeta_1 & -11/7 & \zeta_2 & 0 & \zeta_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 3 & 1 & 0 & 1 & 1 \\ 2 & 4 & 2 & 0 & 2 & 1 \end{array}\right\},$$

where $\zeta_1 < \zeta_2 < \zeta_3$ are the roots of the discriminant $x^3 + 159x^2 + 84x - 1$. The singularities at $x = \zeta_1, \zeta_2, \zeta_3$ are called conifold and there is no monodromy around the point $x = -11/7$, called an apparent singularity.

We expect that the MUM point at $x = \infty$ also have a geometric interpretation and assume all the assumptions in § 3.4. Once passing to a numerical calculation, we obtain the following results.

1. There exists the integral symplectic basis $\Pi^X(x)$ and $z\Pi^Z(z)$ with the parameters,

$$a = -1/2,\ c = -1,\ \deg(Z) = 21,\ c_2(Z) \cdot H = 66,\ \chi(Z) = -102,\ a^Z = -1/2.$$

2. The analytic continuation along a path in the upper half plane gives the following relation of two basis $\Pi^X(x)$ and $z\Pi^Z(z)$,

$$\Pi^X(x) = N_z S_{xz} z \Pi^Z(z),$$

with $N_z = 1$ and the symplectic matrix $S_{xz} = \begin{pmatrix} 8 & 4 & 2 & 5 \\ 4 & 0 & 1 & 2 \\ 10 & -25 & 2 & -1 \\ -21 & 2 & -5 & -10 \end{pmatrix}$.

3. With respect to the local basis of the analytic continuation of $\Pi^X(x)$ and $z\Pi^Z(z)$ along a path in the upper half plane, the monodromy matrices M_p^X and M_p^Z at each singular point $x = -\frac{1}{z} = \zeta_1, \zeta_2, 0, \zeta_3, \infty$ have the following form, respectively:

M_p^X	$\begin{pmatrix} 169 & -80 & 32 & 64 \\ 84 & -39 & 16 & 32 \\ 210 & -100 & 41 & 80 \\ -441 & 210 & -84 & -167 \end{pmatrix}$	$\begin{pmatrix} 13 & -8 & 2 & 4 \\ 6 & -3 & 1 & 2 \\ 24 & -16 & 5 & 8 \\ -36 & 24 & -6 & -11 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 16 & 33 & 1 & 0 \\ -12 & -17 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 286 & -130 & 55 & 111 \\ 89 & -43 & 17 & 34 \\ -307 & 127 & -60 & -122 \\ -465 & 218 & -89 & -179 \end{pmatrix}$
M_p^Z	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -9 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 343 & -17 & 83 & 168 \\ 104 & -9 & 25 & 50 \\ -496 & 8 & -121 & -247 \\ -432 & 32 & -104 & -209 \end{pmatrix}$	$\begin{pmatrix} 211 & -20 & 50 & 100 \\ 105 & -9 & 25 & 50 \\ 42 & -4 & 11 & 20 \\ -441 & 42 & -105 & -209 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 10 & 21 & 1 & 0 \\ -9 & -11 & -1 & 1 \end{pmatrix}$

Table 4.7.1: Monodromy matrices

All in the above results strongly indicate the existence of the geometric interpretation at $x = \infty$. Thus we are led to the following conjecture based on the homological mirror symmetry similar to the examples of the Grassmannian–Pfaffian in [Rød] and the Reye congluence Calabi–Yau 3-fold in [HT1].

Conjecture 4.7.2. *There exists a smooth Calabi–Yau 3-fold Z whose derived category of coherent sheaves is equivalent to that of $X = \mathbf{\Sigma}(1^9)$. The topological invariants of Z are*

$$\deg(Z) = 21, \quad c_2(Z).H = 66, \quad \chi(Z) = -102, \quad h^{1,1}(Z) = 1, \quad h^{2,1}(Z) = 52,$$

where H is the ample generator of the Picard group $\mathrm{Pic}(Z) \simeq \mathbb{Z}$.

The Calabi–Yau 3-fold Z in Conjecture 4.7.2 can not be birational to X because $h^{1,1} = 1$ and $\deg(Z) \neq \deg(X)$, so that it should be a non-trivial Fourier–Mukai partner of X.

5

Complete Intersections of Grassmannians

In this chapter, we study the mirror symmetry for an example $(G(2,5)^2)$.

5.1 Complete intersection of projective varieties

We use the word *complete intersection* in a generalized sense as follows.

Definition 5.1.1. A projective variety $X \subset \mathbb{P}^N$ is called a complete intersection of projective varieties $V_1, \ldots, V_r$ if $X = V_1 \cap \cdots \cap V_r$ as a scheme for some simultaneous embeddings $V_1, \ldots, V_r \subset \mathbb{P}^N$ and $\operatorname{codim} X = \operatorname{codim} V_1 + \cdots + \operatorname{codim} V_r$. We denote by $(V_1, \ldots, V_r)$ a general complete intersection of $V_1, \ldots, V_r$.

We explain an idea of regarding any complete intersection variety in this sense as a complete intersection of hyperplanes in another high dimensional variety. Let $V_1 \subset \mathbb{P}^n_1$ and $V_2 \subset \mathbb{P}^n_2$ be projective varieties in n-dimensional projective subspaces $\mathbb{P}^n_1, \mathbb{P}^n_2 \subset \mathbb{P}^{2n+1}$ with general positions. A choice of an additional general projective subspace $\mathbb{P}^n \subset \mathbb{P}^{2n+1}$ gives an identification $\mathbb{P}^n_1 \simeq \mathbb{P}^n_2$ by regarding it as a graph of the isomorphism. Therefore a complete intersection variety of V_1 and V_2 in $\mathbb{P}^n$ coincides with a complete intersection of a projective subspace $\mathbb{P}^n$ and the projective join $J(V_1, V_2)$ in $\mathbb{P}^{2n+1}$. In particular, we have $(V_1, V_2) \simeq J(V_1, V_2)(1^{n+1})$. Of course, the story can be generalized for complete intersections of $r > 2$ varieties by defining the projective join of r varieties as $J(V_1, V_2, \ldots, V_r) := J(V_1, J(V_2, \ldots, V_r))$.

Example 5.1.2. A general complete intersection of two Grassmannians $G(2,5) \subset \mathbb{P}^9$ is a smooth Calabi–Yau 3-fold of Picard number one [Kan]. We denote by $X = (G(2,5))^2) := (G(2,5), G(2,5)) \simeq J(G(2,5), G(2,5))(1^{10})$ in this section. The topological invariants of X are obtained by [Kan] as

$$\deg X = 25, \quad c_2(X) \cdot H = 70, \quad \chi(X) = -100,$$

where H is the ample generator of the Picard group Pic $X \simeq \mathbb{Z}$.

5.2 Mirror symmetry for $(G(2,5)^2)$

We have a toric degeneration of Grassmannians $G(2,5)$ to the Hibi toric variety $P(2,5) = \mathbb{P}_\Delta(P)$ [BCFKvS1] (Theorem 2.3.2), where the Hasse diagram of the poset P is shaped like a rectangle. From Corollary 2.3.3, we obtain a toric degeneration of $J(G(2,5), G(2,5)) \subset \mathbb{P}^{19}$ to the Hibi toric variety $\mathbb{P}_{\Delta(J(P,P))}$. The Hasse diagram of the projective join $J(P,P)$ is depicted in Figure 1.3.1. Thus we can use all the results in § 3 based on the conjectural mirror construction via conifold transition.

Remark 5.2.1. The Calabi–Yau 3-fold $X = (G(2,5)^2)$ also degenerates to a complete intersection $X_0 = \mathbb{P}_{\Delta(J(P,P))}(1^{10})$ which has $5+5+5+5 = 20$ nodes. Of course, we can recover the topological invariants for X in Example 5.1.2 using the procedure in § 4.6.

The fundamental period of the conjectural mirror family of X can be read from the diagram in Figure 1.3.1 and turns out to be

$$\begin{aligned} \omega_0(x) &= \sum_{m,s,t,u,v} \binom{m}{s}\binom{m}{t}^2\binom{t}{s}\binom{m}{u}\binom{m}{v}^2\binom{v}{u} x^m \\ &= \sum_m \left\{ \sum_{s,t} \binom{m}{s}\binom{m}{t}^2\binom{t}{s} \right\}^2 x^m. \end{aligned} \tag{5.2.1}$$

where $x = a^{10}$. From this power series expansion, we obtain the Picard–Fuchs equation for the conjectural mirror family of X.

Proposition 5.2.2. *Let $\omega_0(x)$ be the above power series around $x = 0$, which corresponds to the fundamental period for the conjectural mirror family of the Calabi–Yau 3-fold $X = (G(2,5)^2)$. This satisfies the Picard–Fuchs equation $\mathcal{D}_x\omega_0(x) = 0$ with $\theta_x = x\partial_x$ and*

$$\begin{aligned} \mathcal{D}_x =& \theta_x^4 - x(124\theta_x^4 + 242\theta_x^3 + 187\theta_x^2 + 66\theta_x + 9) + \\ &+ x^2(123\theta_x^4 - 246\theta_x^3 - 787\theta_x^2 - 554\theta_x - 124) + \\ &+ x^3(123\theta_x^4 + 738\theta_x^3 + 689\theta_x^2 + 210\theta_x + 12) - \\ &- x^4(124\theta_x^4 + 254\theta_x^3 + 205\theta_x^2 + 78\theta_x + 12) + x^5(\theta_x + 1)^4. \end{aligned}$$

The Riemann scheme of the differential operator $\mathcal{D}_x$ is

$$P\left\{\begin{array}{cccccc} -1 & 0 & \zeta_1 & 1 & \zeta_2 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array}\right\},$$

where $\zeta_1 < \zeta_2$ are the roots of the discriminant $x^2 - 123x + 1$. The singularities at $x = -1, \zeta_1, \zeta_2$ are called conifold (although -1 is slightly different from the latter two, conjecturally the point where a lens space $S^3/\mathbb{Z}_2$ vanishes), and the point $x = 1$ is an apparent singularity.

The MUM point at $x = \infty$ seems to correspond to the same geometry as around $x = 0$ because of the operator identity $z\mathcal{D}_{1/z}z^{-1} = \mathcal{D}_z$. We may assume all the assumptions in § 3.4 together with $Z = X = (G(2,5)^2)$. Then we obtain the following results.

1. There exists the integral symplectic basis $\Pi^X(x)$ and $z\Pi^X(z)$ with the parameters,

$$a = -1/2, \qquad c = 1.$$

2. The analytic continuation along a path in the upper half plane gives the following relation of two basis $\Pi^X(x)$ and $z\Pi^X(z)$,

$$\Pi^X(x) = N_z S_{xz} z\Pi^X(z),$$

with $N_z = 1$ and the symplectic matrix $S_{xz} = \begin{pmatrix} -4 & 7 & -1 & 4 \\ 0 & 4 & 0 & 1 \\ 15 & 0 & 4 & -8 \\ 0 & -15 & 0 & -4 \end{pmatrix}$.

3. With respect to the local basis of the analytic continuation of $\Pi^X(x)$ and $z\Pi^X(z)$ along a path in the upper half plane, the monodromy matrices M_p^X at each singular point $x = \frac{1}{z} = -1, 0, \zeta_1, \zeta_2, \infty$ have the following form:

M_p^X	$\begin{pmatrix} 21 & -8 & 4 & 8 \\ 10 & -3 & 2 & 4 \\ 20 & -8 & 5 & 8 \\ -50 & 20 & -10 & -19 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 12 & 25 & 1 & 0 \\ -10 & -13 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 60 & 0 & 16 \\ 0 & 1 & 0 & 0 \\ 0 & -225 & 1 & -60 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -19 & 248 & -4 & 75 \\ 9 & -93 & 2 & -29 \\ -17 & -118 & -5 & -23 \\ -40 & 383 & -9 & 121 \end{pmatrix}$

Table 5.2.1: Monodromy matrices

Let $V \subset \mathbb{P}^n$ be a smooth projective variety of Picard number one. In [BCFKvS1], the authors introduce the A-series $A_V(q)$ of V as a holomorphic power series solution of the quantum cohomology $\mathcal{D}$-module of V. Let $X = V(d_1, \ldots, d_r) \subset \mathbb{P}^n$ be a Calabi–Yau complete intersection of V and general hypersurfaces in $\mathbb{P}^n$ of degree d_j ($j = 1, \ldots, r$). They say that the *Trick with Factorials works* for X if the fundamental period $\omega_0(x)$ of the mirror family X^* is written as

$$\omega_0(x) = \sum_{m=0}^{\infty} a_m^V (md_1)! \cdots (md_r)! x^m, \tag{5.2.2}$$

where $A_V = \sum_{m=0}^{\infty} a_m^V q^m$. The Trick with Factorials is a special version of the quantum hyperplane section theorem [Kim].

It seems natural to generalize their definition as follows:

Definition 5.2.3. Let $V_1, \cdots, V_r \subset \mathbb{P}^n$ be smooth Fano manifolds of Picard number one and $X = (V_1, \cdots, V_r) \subset \mathbb{P}^n$ be a general Calabi–Yau complete intersection of Fano manifolds V_j ($j = 1, \ldots, r$). We say that the *Trick with Factorials works* for X if the fundamental period $\omega_0(x)$ of the mirror family X^* is written as

$$\omega_0(x) = \sum_{m=0}^{\infty} a_m^{V_1} \cdots a_m^{V_r} x^m, \tag{5.2.3}$$

where $A_{V_j} = \sum_{m=0}^{\infty} a_m^{V_j} q^m$ ($j = 1, \ldots, r$).

From the formula (5.2.1) of the fundamental period, we expect the following.

Conjecture 5.2.4. *The Trick with Factorials works for* $(G(2,5)^2)$.

A
BPS numbers

As a further consistency check in Conjecture 4.7.2 or an application of the mirror construction, we carry out the computation of BPS numbers by using the methods proposed by [CdOGP] [BCOV1, BCOV2]. The BPS numbers $n_g(d)$ are related with the Gromov–Witten invariants $N_g(d)$ by the following formula [GV],

$$\sum_{g\geq 0} N_g(d)\lambda^{2g-2} = \sum_{k|d}\sum_{g\geq 0} n_g(d/k)\frac{1}{k}(2\sin\frac{k\lambda}{2})^{2g-2}.$$

Hence we obtain the prediction for Gromov–Witten invariants from the computations.

We skip all the details and only present results here for $X = \mathbf{\Sigma}(1^9)$ and its conjectural Fourier–Mukai partner Z and $(G(2,5)^2)$. For the details, one can get many references in now. Here we have followed [HK], where a very similar example to ours, the Grassmannian–Pfaffian Calabi–Yau 3-fold, has been analyzed.

A.1 BPS numbers for $X = \mathbf{\Sigma}(1^9)$ and Z

d	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$
1	252	0	0	0	0
2	1854	0	0	0	0
3	27156	0	0	0	0
4	567063	0	0	0	0
5	14514039	4671	0	0	0
6	424256409	1029484	0	0	0
7	13599543618	112256550	5058	0	0
8	466563312360	9161698059	7759089	0	0
9	16861067232735	645270182913	2496748119	151479	0
10	634912711612848	41731465395267	438543955881	418482990	-3708
11	24717672325914858	2557583730349461	56118708041940	217285861284	33975180

Table A.1.1: BPS numbers $n_g^X(d)$ of $X = \mathbf{\Sigma}(1^9)$

d	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$
1	387	0	0	0	0
2	4671	0	0	0	0
3	124323	1	0	0	0
4	4782996	1854	0	0	0
5	226411803	606294	0	0	0
6	12249769449	117751416	27156	0	0
7	727224033330	17516315259	33487812	252	0
8	46217599569117	2252199216735	15885697536	7759089	0
9	3094575464496057	265984028638047	4690774243470	13680891072	1127008
10	215917815744645750	29788858876065588	1053460470463461	9429360817149	12259161360

Table A.1.2: BPS numbers $n_g^Z(d)$ of Z

A.2 BPS numbers for $X = (G(2,5)^2)$

d	$g = 0$	$g = 1$	$g = 2$	$g = 3$
1	325	0	0	0
2	3200	0	0	0
3	66250	0	0	0
4	1985000	325	0	0
5	73034875	109822	0	0
6	3070310300	19018900	650	0
7	141603560675	2367994150	1829200	0
8	6990803723200	247337794725	938148600	72650
9	363591194115575	23368078640700	253848387875	287055600
10	19705196405545000	2075562931676048	48865015050900	225293359750
11	1104153966524594850	177059059777938850	7643658178867550	90644383230350
12	63598129792406485600	14692505162221545750	1041954995886347300	25018039373344450

Table A.2.1: BPS numbers $n_g^X(d)$ of $X = Z = (G(2,5)^2)$

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